A Pontryagin approach to delegation problems

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The (cumulative) Lagrangian methods developed by Amador et al. (2006) have been widely used to solve the maximization problems in delegation (e.g. Amador and Bagwell, 2013, 2022; Guo, 2016). In this note, I apply Pontryagin's maximum principles in the optimal control theory to solve these problems, as a less algebra-intensive alternative. Moreover, in delegation problems with participation constraints (Amador and Bagwell, 2022), the hybrid maximum principles make it possible to study the global problem that involves a jump in the allocation at the cutoff type (due to the participation constraint), instead of the truncated problem at the cutoff, and thus provide weaker sufficient conditions that are also necessary.

1 Without participation constraint (Amador and Bagwell, 2013)

I first revisit the delegation problem in Amador and Bagwell (2013) and show that the Pontryagin approach can be applied to solve the problem. The maximization problem in Amador and Bagwell (2013) is

$$\max_{\pi:\Gamma\to\Pi} \int w(\gamma,\pi(\gamma))\,\mathrm{d}F(\gamma) \quad \text{subject to:} \tag{1}$$

$$\gamma\pi(\gamma)+b(\pi(\gamma)) = \int_{\underline{\gamma}}^{\gamma} \pi(\tilde{\gamma})d\tilde{\gamma}+\underline{U} \quad \text{for all } \gamma\in\Gamma, \tag{2}$$

$$\pi \text{ nondecreasing,} \tag{3}$$

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where $\underline{U} \equiv \underline{\gamma}\pi(\underline{\gamma}) + b(\pi(\underline{\gamma}))$.[1] Define $U = \int_{\underline{\gamma}}^{\gamma} \pi(\tilde{\gamma})d\tilde{\gamma} + \underline{U}$. Rewrite the constraints as

$$\gamma\pi(\gamma) + b(\pi(\gamma)) = U \tag{4}$$
$$\dot{U} = \pi \tag{5}$$
$$\dot{\pi} = \nu \geq 0 \tag{6}$$
$$U(\underline{\gamma}), \pi(\underline{\gamma}) \text{ free}, \tag{7}$$
$$U(\bar{\gamma}), \pi(\bar{\gamma}) \text{ free} \tag{8}$$

Set up the Hamiltonian:

$$H = w(\gamma, \pi)f(\gamma) + \lambda(\gamma\pi(\gamma) + b(\pi(\gamma)) - U) + \Lambda\pi + \mu\nu \tag{9}$$

where π, U are the state variable and ν is the control variable; Λ is the Hamiltonian multiplier on $\dot{U}$, μ is the Hamiltonian multiplier on $\dot{q}$, and λ is the Lagrangian multiplier on $\gamma\pi(\gamma) + b(\pi(\gamma)) = U$.

By the Pontryagin's maximum principle,

$$-\frac{\partial H}{\partial \pi} = -(w_\pi f + \lambda(\gamma + b'(\pi)) + \Lambda) = \dot{\mu} \tag{10}$$
$$-\frac{\partial H}{\partial U} = \lambda = \dot{\Lambda} \tag{11}$$
$$\frac{\partial H}{\partial \nu} = \mu \leq 0, \quad \mu\nu = 0 \tag{12}$$
$$\Lambda(\underline{\gamma}) = 0, \quad \mu(\underline{\gamma}) = 0 \tag{13}$$
$$\Lambda(\bar{\gamma}) = 0, \quad \mu(\bar{\gamma}) = 0 \tag{14}$$

Define $\pi_f(\gamma) = \arg\max_\pi\{\gamma\pi + b(\pi)\}$ as the flexible allocation. For the following interval delegation to be optimal,

$$\pi(\gamma) = \begin{cases} \pi_f(\gamma_L); & \gamma \in [\gamma, \gamma_L], \\ \pi_f(\gamma); & \gamma \in (\gamma_L, \gamma_H), \\ \pi_f(\gamma_H); & \gamma \in [\gamma_H, \bar{\gamma}] \end{cases} \tag{15}$$

[1]Note that the standard method is not applicable because transfers are infeasible.

the proposed multipliers are

$$\Lambda(\gamma)=\begin{cases}\kappa(1-F(\gamma)), & \gamma\in[\gamma_H,\bar{\gamma}]\,,\\ -w_\pi\left(\gamma,\pi_f(\gamma)\right)f(\gamma), & \gamma\in(\gamma_L,\gamma_H)\,,\\ -\kappa F(\gamma), & \gamma\in\left[\underline{\gamma},\gamma_L\right]\end{cases}\tag{16}$$

$$\mu(\gamma)=\begin{cases}\int_\gamma^{\bar{\gamma}}(w_\pi(\tilde{\gamma},\pi_f(\gamma_H))f(\tilde{\gamma})-\kappa\tilde{\gamma}f(\tilde{\gamma})+\kappa(1-F(\tilde{\gamma})))\,\mathrm{d}\tilde{\gamma}\le 0, & \gamma\in[\gamma_H,\bar{\gamma}]\,,\\ 0, & \gamma\in(\gamma_L,\gamma_H)\,,\\ -\int_{\underline{\gamma}}^{\gamma}(w_\pi(\tilde{\gamma},\pi_f(\gamma_L))f(\tilde{\gamma})-\kappa\tilde{\gamma}f(\tilde{\gamma})+\kappa F(\tilde{\gamma}))\,\mathrm{d}\tilde{\gamma}\le 0, & \gamma\in\left[\underline{\gamma},\gamma_L\right]\end{cases}\tag{17}$$

The inequality $\mu(\gamma)\le 0$ follows from their assumption (c2) [resp. (c3)] if γ_H [resp. γ_L] is interior. If γ_H or γ_L is at the boundary, the corresponding inequality is satisfied trivially.

Sufficiency. By Kamien and Schwartz (1971), the sufficient condition is that the maximized Hamiltonian $\bar{H}(\pi,U,\Lambda,\mu,\lambda)\equiv\max_\nu H(\pi,U,\nu,\Lambda,\mu,\lambda)$ is concave in (π,U) for given (Λ,μ,λ), which requires $w_{\pi\pi}f+\lambda b''(\pi)\le 0$. Following the definition of $\kappa=\inf_{\pi,\gamma}\{w_{\pi\pi}/b''(\pi)\}$, concavity is satisfied if $\Lambda+\kappa F$ is increasing. Therefore, their assumption (c1): $\kappa F(\gamma)-w_\pi\left(\gamma,\pi_f(\gamma)\right)f(\gamma)$ is nondecreasing for all $\gamma\in[\gamma_L,\gamma_H]$ guarantees concavity on (γ_L,γ_H), and (c2') and (c3') guarantees increasing at $\underline{\gamma}$ and $\bar{\gamma}$, respectively (if γ_H or γ_L are at the boundary).

2 With participation constraint (Amador and Bagwell, 2022)

Amador and Bagwell (2022, henceforth AB) add a participation constraint to the maximization program, which makes the problem trickier because the state variable q (i.e., π in Amador and Bagwell, 2013) can have jumps at the cutoff type γ_t. Therefore, they study the truncated problem at the cutoff instead. Nevertheless, the untruncated problem can still be solved by optimal control by invoking the maximum principle in Hellwig (2010, Theorem 4.1) for the conditions of the Hamiltonian multiplier on $\dot{q}$ at discontinuities of the increasing q and the hybrid maximum principle in Bryson and Ho (1975, Chapter 3.7) for the jump (switching) conditions at γ_t (see also Clarke, 2013, Chapter 22.5).

2.1 The truncated problem in Amador and Bagwell (2022)

Amador and Bagwell (2022) study the following truncated problem (P_t) at the cutoff type γ_t. Before applying the hybrid maximum principle to the untruncated problem, I replicate their result for the truncated problem.

$$\max_{q_t:\Gamma_t(\gamma_t)\to Q} \int_{\Gamma_t(\gamma_t)} \left(w\left(\gamma, q_t(\gamma)\right) - \sigma\right) dF(\gamma) \quad \text{subject to:} \tag{18}$$

$$-\gamma q_t(\gamma) + b\left(q_t(\gamma)\right) - \sigma - \int_{\gamma}^{\gamma_t} q_t(\tilde{\gamma}) d\tilde{\gamma} = \bar{U}, \quad \text{for all } \gamma \in \Gamma_t(\gamma_t) \tag{19}$$

$$q_t(\gamma) \text{ decreasing}, \quad \text{for all } \gamma \in \Gamma_t(\gamma_t) \tag{20}$$

$$0 \le -\gamma q_t(\gamma) + b\left(q_t(\gamma)\right) - \sigma, \quad \text{for all } \gamma \in \Gamma_t(\gamma_t), \tag{21}$$

where $\bar{U} \equiv -\gamma_t q_t(\gamma_t) + b\left(q_t(\gamma_t)\right) - \sigma$. Define $U(\gamma) = \int_{\gamma}^{\gamma_t} q(x)dx + \bar{U}$. Rewrite the constraints as:[2]

$$U(\gamma) = -\gamma q_t(\gamma) + b(q_t(\gamma)) - \sigma \tag{22}$$

$$\dot{U} = -q_t(\gamma) \tag{23}$$

$$\dot{q}_t = \nu(\gamma) \le 0 \tag{24}$$

$$U(\gamma_t), q_t(\gamma_t) \ge 0 \tag{25}$$

$$U(\underline{\gamma}), q_t(\underline{\gamma}) \text{ free.} \tag{26}$$

Set up the Hamiltonian

$$H = [w(q_t(\gamma), \gamma) - \sigma] f(\gamma) + \lambda(\gamma)[-\gamma q_t(\gamma) + b(q_t(\gamma)) - U(\gamma) - \sigma] - \Lambda(\gamma) q_t(\gamma) + \mu(\gamma)\nu(\gamma) \tag{27}$$

where U, q are state variables and ν is the control variable; Λ is Hamiltonian multiplier on $\dot{U}$ and μ is Hamiltonian multiplier on $\dot{q}$; λ is the Lagrangian multiplier on $U = -\gamma q_t + b(q_t)$.

[2](IR) is not explicitly written because it is implied by $U(\gamma_t), q_t(\gamma_t) \ge 0$ and $\dot{U} = -q_t(\gamma)$.

By the Pontryagin's maximum principle,

$$-\frac{\partial H}{\partial q} = -w_q f + \lambda(\gamma - b'(q)) + \Lambda = \dot{\mu} \tag{28}$$

$$-\frac{\partial H}{\partial U} = \lambda = \dot{\Lambda} \tag{29}$$

$$\frac{\partial H}{\partial \nu} = \mu \geq 0, \quad \mu\nu = 0 \tag{30}$$

$$\Lambda(\gamma_t) \leq 0, \quad \Lambda(\gamma_t)U(\gamma_t) = 0 \tag{31}$$

$$\mu(\gamma_t) \leq 0, \quad \mu(\gamma_t)q(\gamma_t) = 0 \tag{32}$$

$$\Lambda(\underline{\gamma}) = 0, \quad \mu(\underline{\gamma}) = 0. \tag{33}$$

Define $q_f(\gamma) = \arg\max_q\{b(q) - \gamma q\}$ as the flexible allocation, and denote by $q_i\left(\gamma_t\right) = \{q : -\gamma_t q + b(q) = \sigma\}$ the quality level at which an agent is indifferent between producing or not. For the following price cap (quantity floor) allocation to be optimal,

$$q_t^\star\left(\gamma \mid \gamma_t\right) = \begin{cases} q_f(\gamma); & \gamma \in [\underline{\gamma}, \gamma_H(\gamma_t)) \\ q_i(\gamma_t); & \gamma \in [\gamma_H(\gamma_t), \gamma_t] \end{cases} \tag{34}$$

the proposed multipliers are

$$\Lambda(\gamma) = \begin{cases} 0; & \gamma = \underline{\gamma} \\ w_q\left(\gamma, q_f(\gamma)\right) f(\gamma); & \gamma \in \left(\underline{\gamma}, \gamma_H(\gamma_t)\right) \\ A + \kappa\left(F(\gamma_t) - F(\gamma)\right); & \gamma \in [\gamma_H(\gamma_t), \gamma_t] \end{cases} \tag{35}$$

where $\kappa \equiv \inf_q\{w_{qq}/b''(q)\}$,

$$A = \frac{1}{\gamma_t - \gamma_H(\gamma_t)}\left[\int_{\gamma_H(\gamma_t)}^{\gamma_t} w_q\left(\gamma, q_i(\gamma_t)\right) f(\gamma) d\gamma + \kappa\left(\gamma_H(\gamma_t) - b'\left(q_i(\gamma_t)\right)\right) F(\gamma_t)\right] \geq 0$$

as proposed by AB.[3]

$$\mu(\gamma)=\begin{cases}\int_{\gamma}^{\gamma_t} w_q(\gamma,q_i(\gamma_t))f(\gamma)\,\mathrm{d}\gamma+\kappa(\gamma-b'(q_i(\gamma_t)))(F(\gamma_t)-F(\gamma))-(\gamma_t-\gamma)A\geq 0, & \gamma\in[\gamma_H(\gamma_t),\gamma_t]\\ 0, & \gamma\in[\underline{\gamma},\gamma_H(\gamma_t)]\end{cases} \tag{36}$$

The inequality $\mu(\gamma)\geq 0$ follows from their condition (i) in Proposition 1.[4]

Sufficiency. The concavity of the maximized Hamiltonian requires $\Lambda+\kappa F$ to be increasing. Their condition (ii) in Proposition 1 guarantees concavity on $(\underline{\gamma},\gamma_H(\gamma_t))$; condition (i) guarantees concavity at $\gamma_H(\gamma_t)$ if it is interior (by $A\geq 0$ if $\gamma_H(\gamma_t)=\underline{\gamma}$); and $w_q>0$ guarantees concavity at $\underline{\gamma}$.

2.2 Untruncated problem

Assuming fixed cost $\sigma=0$, the untruncated problem (P) is given by

$$\max_{q:\Gamma\to Q}\int_{\Gamma}\left(w\left(\gamma,q(\gamma)\right)-\sigma\mathbf{1}(q(\gamma))\right)dF(\gamma)\quad\text{subject to:} \tag{38}$$

$$-\gamma q(\gamma)+b\left(q(\gamma)\right)-\sigma\mathbf{1}(q(\gamma))-\int_{\gamma}^{\bar{\gamma}}q(\tilde{\gamma})\,\mathrm{d}\tilde{\gamma}=\bar{U},\quad\text{for all }\gamma\in\Gamma \tag{39}$$

$$q(\gamma)\text{ decreasing},\quad\text{for all }\gamma\in\Gamma \tag{40}$$

$$0\leq-\gamma q(\gamma)+b\left(q(\gamma)\right)-\sigma\mathbf{1}(q(\gamma)),\quad\text{for all }\gamma\in\Gamma, \tag{41}$$

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[3] In the next subsection, I will propose

$$\hat{A}=\frac{1}{\gamma_t-b'(q_i(\gamma_t))}\int_{\gamma_H(\gamma_t)}^{\gamma_t}w_q\left(\gamma,q_i(\gamma_t)\right)\mathrm{d}F(\gamma)\in[0,A],$$

which is also applicable here.

[4] They define

$$G\left(\gamma|\gamma_t\right)\equiv-\kappa F(\gamma_t)+\kappa\left[\frac{\gamma-b'\left(q_i(\gamma_t)\right)}{\gamma-\gamma_H(\gamma_t)}\right]F(\gamma)+\frac{1}{\gamma-\gamma_H(\gamma_t)}\int_{\gamma_H(\gamma_t)}^{\gamma}w_q\left(\tilde{\gamma},q_i(\gamma_t)\right)f(\tilde{\gamma})d\tilde{\gamma} \tag{37}$$

and assume

Condition (i). $G\left(\gamma|\gamma_t\right)\leq G\left(\gamma_t\mid\gamma_t\right)$ for all $\gamma\in[\gamma_H(\gamma_t),\gamma_t]$.

Condition (ii). $M_1(\gamma)\equiv\kappa F(\gamma)+w_q\left(\gamma,q_f(\gamma)\right)f(\gamma)$ is nondecreasing in γ for $\gamma\in[\underline{\gamma},\gamma_H(\gamma_t))$.

where $\bar{U} \equiv -\gamma q(\gamma) + b(q(\gamma)) - \sigma$. Define $U(\gamma) = \int_{\gamma}^{\bar{\gamma}} q(x)dx + \bar{U}$. Rewrite the constraints as:

$$U(\gamma) = -\gamma q(\gamma) + b(q(\gamma)) - \sigma \mathbf{1}(q(\gamma)) \tag{42}$$
$$\dot{U} = -q(\gamma) \tag{43}$$
$$\dot{q} = \nu(\gamma) \leq 0 \tag{44}$$
$$U(\gamma), q(\gamma) \geq 0 \tag{45}$$
$$U(\underline{\gamma}), q(\underline{\gamma}) \text{ free.} \tag{46}$$

Set up the Hamiltonian

$$\begin{aligned} H = [w(q(\gamma), \gamma) - \sigma \mathbf{1}(q(\gamma))] f(\gamma) + \lambda(\gamma)[-\gamma q(\gamma) + b(q(\gamma)) - U(\gamma) - \sigma \mathbf{1}(q(\gamma))] \\ -\Lambda(\gamma) q(\gamma) + \mu(\gamma)\nu(\gamma). \end{aligned} \tag{47}$$

By the Pontryagin's maximum principle (Hellwig, 2010, Theorem 4.1),

$$-\frac{\partial H}{\partial q} = -w_q f + \lambda(\gamma - b'(q)) + \Lambda = \dot{\mu} \tag{48}$$
$$-\frac{\partial H}{\partial U} = \lambda = \dot{\Lambda} \tag{49}$$
$$\frac{\partial H}{\partial \nu} = \mu \geq 0, \quad \mu(\gamma) = 0 \text{ if } q \text{ is strictly decreasing at } \gamma \tag{50}$$
$$\Lambda(\gamma_t) \leq 0, \quad \Lambda(\gamma_t) U(\gamma_t) = 0 \tag{51}$$
$$\mu(\gamma_t) \leq 0, \quad \mu(\gamma_t) q(\gamma_t) = 0 \tag{52}$$
$$\Lambda(\underline{\gamma}) = 0, \quad \mu(\underline{\gamma}) = 0. \tag{53}$$

By the jump (switching) condition at γ_t (Bryson and Ho, 1975, Chapter 3.7) (see also Clarke, 2013, Chapter 22.5 for the hybrid maximum principle),

$$\Lambda(\gamma_t -) = \Lambda(\gamma_t +) \tag{54}$$

$$H(\gamma_t -) = (w(q_t, \gamma_t) - \sigma) f(\gamma_t) - \Lambda(\gamma_t -) q_t = H(\gamma_t +) = 0. \tag{55}$$

For the following price cap (quantity floor) allocation to be optimal,

$$q_t^{\star}(\gamma) = \begin{cases} q_f(\gamma); & \gamma \in [\underline{\gamma}, \gamma_H(\gamma_t)) \\ q_i(\gamma_t); & \gamma \in [\gamma_H(\gamma_t), \gamma_t] \\ 0; & \gamma \in (\gamma_t, \bar{\gamma}] \end{cases} \tag{56}$$

the proposed multipliers are

$$\Lambda(\gamma) = \begin{cases} 0; & \gamma = \underline{\gamma} \\ w_q\left(\gamma, q_f(\gamma)\right) f(\gamma); & \gamma \in \left(\underline{\gamma}, \gamma_H(\gamma_t)\right) \\ \hat{A} + \kappa\left(F(\gamma_t) - F(\gamma)\right); & \gamma \in [\gamma_H(\gamma_t), \bar{\gamma}] \end{cases} \tag{57}$$

where $\kappa \equiv \inf_q\{w_{qq}/b''(q)\}$,[5]

$$\hat{A} = \frac{1}{\gamma_t - b'(q_i(\gamma_t))} \int_{\gamma_H(\gamma_t)}^{\gamma_t} w_q\left(\gamma, q_i(\gamma_t)\right) \mathrm{d}F(\gamma) \geq 0,$$

and the cutoff type γ_t, determined by the jump condition (55), is given by

$$(w(\gamma_t, q_i(\gamma_t)) - \sigma) f(\gamma_t) - \hat{A} q_i(\gamma_t) \geq 0, \quad [(w(\gamma_t, q_i(\gamma_t)) - \sigma) f(\gamma_t) - \hat{A} q_i(\gamma_t)](\bar{\gamma} - \gamma_t) = 0. \tag{58}$$

In particular, if $(w(\gamma_t, q_i(\gamma_t)) - \sigma) f(\gamma_t) - \hat{A} q_i(\gamma_t) > 0$, the cutoff type is $\gamma_t = \bar{\gamma}$.

$$\mu(\gamma) = \begin{cases} -\int_{\gamma_t}^{\gamma} w_q(\tilde{\gamma}, 0) f(\tilde{\gamma})\, \mathrm{d}\tilde{\gamma} - \kappa\gamma(F(\gamma) - F(\gamma_t)) + (\gamma - \gamma_t)\hat{A} \geq 0, & \gamma \in [\gamma_t, \bar{\gamma}] \\ \int_{\gamma}^{\gamma_t} w_q(\tilde{\gamma}, q_i(\gamma_t)) f(\tilde{\gamma})\, \mathrm{d}\tilde{\gamma} + \kappa(\gamma - b'(q_i(\gamma_t)))(F(\gamma_t) - F(\gamma)) - (\gamma_t - \gamma)\hat{A} \geq 0, & \gamma \in [\gamma_H(\gamma_t), \gamma_t] \\ 0, & \gamma \in \left[\underline{\gamma}, \gamma_H(\gamma_t)\right] \end{cases} \tag{59}$$

The inequality $\mu(\gamma) \geq 0$ requires conditions analogous to their condition (i) in Proposition 1.

2.3 Weaker Conditions

Define

$$s(\gamma|q) = w_q(\gamma, q) f(\gamma) + \kappa f(\gamma)(\gamma - b'(q)) + \kappa(F(\gamma) - F(\gamma_t)) \tag{60}$$

and, slightly abusing notations,

$$s(\gamma) = s(\gamma|q^*) = \begin{cases} w_q(\gamma, 0) f(\gamma) + \kappa f(\gamma)(\gamma - b'(0)) + \kappa(F(\gamma) - F(\gamma_t)), & \text{if } \gamma \in (\gamma_t, \bar{\gamma}] \\ w_q(\gamma, q_i(\gamma_t)) f(\gamma) + \kappa f(\gamma)(\gamma - b'(q_i(\gamma_t))) + \kappa(F(\gamma) - F(\gamma_t)), & \text{if } \gamma \in [b'(q_i(\gamma_t)), \gamma_t) \end{cases} \tag{61}$$

[5] By convention, $F(\gamma) = 0$ for all $\gamma \leq \underline{\gamma}$.

and $S(\gamma) = \int_{\underline{\gamma}}^{\gamma} s(\tilde{\gamma})\,\mathrm{d}\tilde{\gamma}$. Define

$$L(\gamma|\gamma_t) = \frac{S(\gamma_t) - S(\gamma)}{\gamma_t - \gamma} = \frac{1}{\gamma_t - \gamma}\int_{\gamma}^{\gamma_t} s(\tilde{\gamma})\,\mathrm{d}\tilde{\gamma} \tag{62}$$

Equivalently,

$$L(\gamma|\gamma_t) = \begin{cases} \frac{1}{\gamma-\gamma_t}\left[\int_{\gamma_t}^{\gamma} w_q(\tilde{\gamma},0)f(\tilde{\gamma})\,\mathrm{d}\tilde{\gamma} + \kappa(\gamma - b'(0))(F(\gamma)-F(\gamma_t))\right], & \text{if } \gamma \in (\gamma_t, \bar{\gamma}] \\ \frac{1}{\gamma_t-\gamma}\left[\int_{\gamma}^{\gamma_t} w_q(\tilde{\gamma},q_i(\gamma_t))f(\tilde{\gamma})\,\mathrm{d}\tilde{\gamma} + \kappa(\gamma - b'(q_i(\gamma_t)))(F(\gamma_t)-F(\gamma))\right], & \text{if } \gamma \in [b'(q_i(\gamma_t)), \gamma_t). \end{cases}$$

In particular, the multiplier $\hat{A} = L(b'(q_i(\gamma_t))|\gamma_t)$, while the multiplier originally proposed by AB is $A \equiv G(\gamma_t|\gamma_t) = L(\gamma_H(\gamma_t)|\gamma_t)$. Because $\gamma_H(\gamma_t) = \max\{b'(q_i(\gamma_t)), \underline{\gamma}\}$, we have $L(b'(q_i(\gamma_t))|\gamma_t) \geq L(\gamma_H(\gamma_t)|\gamma_t)$, where the equality holds if and only if $b'(q_i(\gamma_t)) \geq \bar{\gamma}$ (so that $b'(q_i(\gamma_t)) = \gamma_H$).

Lemma 1. $L(\gamma_t+|\gamma_t) \geq L(\gamma_t-|\gamma_t)$ *for all* $\gamma_t \in (\underline{\gamma}, \bar{\gamma})$. *The equality holds if and only if* $w_{qq}(q,\gamma_t) + \kappa b''(q) = 0$ *for almost every* $q \in (0, q_i(\gamma_t))$.

Now I propose two weaker conditions on $L(\gamma|\gamma_t)$ that support both the pooling and the exclusion region, respectively.

Condition (i'). $L(\gamma|\gamma_t) \geq L(b'(q_i(\gamma_t))|\gamma_t) = \hat{A}$ for all $\gamma \in [\gamma_H(\gamma_t), \gamma_t)$.

Condition (iii). $L(\gamma|\gamma_t) \leq L(b'(q_i(\gamma_t))|\gamma_t) = \hat{A}$ for all $\gamma \in (\gamma_t, \bar{\gamma}]$.

Graphical interpretations. Graphically, the conditions mean the line ℓ connecting γ_t and $b'(q_i(\gamma_t))$ on $S(\theta)$ (which has a slope of $L(b'(q_i(\gamma_t))|\gamma_t) = \hat{A}$) does not intersect with S at any $\gamma \in (\gamma_H(\gamma_t), \bar{\gamma})$ other than θ_t. In other words, k is the supporting hyperplane (line) of the epigraph of $S(\gamma)$ on $\gamma \in [\gamma_H(\gamma_t), \bar{\gamma}]$ containing γ_t. As noted in Observation 1, if $\gamma_t < \bar{\gamma}$, Lemma 1 implies that ℓ must be tangent to $S(\gamma)$ at γ_t.

Observation 1. If $\gamma_t < \bar{\gamma}$, then conditions (i') and (iii) imply $L(\gamma_t+|\gamma_t) \leq L(\gamma_t-|\gamma_t)$. Therefore, by Lemma 1, it must be $L(\gamma_t+|\gamma_t) = L(\gamma_t-|\gamma_t) = \hat{A}$.

The two conditions can be combined into a single condition that involves $s(\gamma)$:

Condition (I). $\int_{\gamma_t}^{\gamma} s(\gamma)\,\mathrm{d}\gamma \leq \hat{A}\cdot(\gamma - \gamma_t)$ for all $\gamma \in [\gamma_H(\gamma_t), \bar{\gamma})$.

If $\gamma_t < \bar{\gamma}$, it becomes:

$\int_{\gamma_t}^{\gamma} s(\tilde{\gamma})\,\mathrm{d}\tilde{\gamma} \leq s(\gamma_t)(\gamma - \gamma_t)$ for all $\gamma \in [\gamma_H(\gamma_t), \bar{\gamma})$, with equality at $\gamma = b'(q_i(\gamma_t))$.

Condition (ii) in AB is still required to guarantee concavity.

Condition (ii). $\kappa F(\gamma) + w_q\left(\gamma, q_f(\gamma)\right) f(\gamma)$ is nondecreasing in γ for $\gamma \in [\underline{\gamma}, \gamma_H(\gamma_t))$.

Proposition 1. *If conditions (i'), (ii) and (iii) hold at some $\gamma_t \in [\underline{\gamma}, \bar{\gamma}]$, then the price cap allocation with cutoff γ_t is optimal.*

Remark 1. If conditions (i') and (iii) hold at some $\gamma_t \in [\underline{\gamma}, \bar{\gamma}]$ such that $b'(q_i(\gamma_t)) \leq \underline{\gamma}$, a bang-bang allocation where firms either exit or set the price at the price cap is optimal.

It is convenient to extend $s(\gamma)$ to $\gamma \in [b'(q_i(\gamma_t)), \bar{\gamma}]$. Because $\gamma_H(\gamma_t) = \max\{b'(q_i(\gamma_t)), \underline{\gamma}\}$, if $b'(q_i(\gamma_t)) \geq \underline{\gamma}$, then $\gamma_H(\gamma_t) = b'(q_i(\gamma_t))$; otherwise, $\gamma_H(\gamma_t) = \underline{\gamma}$.

$$s(\gamma) = s(\gamma|q^*) = \begin{cases} w_q(\gamma, 0)f(\gamma) + \kappa f(\gamma)(\gamma - b'(0)) + \kappa(F(\gamma) - F(\gamma_t)), & \text{if } \gamma \in (\gamma_t, \bar{\gamma}] \\ w_q(\gamma, q_i(\gamma_t))f(\gamma) + \kappa f(\gamma)(\gamma - b'(q_i(\gamma_t))) + \kappa(F(\gamma) - F(\gamma_t)), & \text{if } \gamma \in [b'(q_i(\gamma_t)), \gamma_t) \\ w_q\left(\gamma, q_f(\gamma)\right) f(\gamma) + \kappa(F(\gamma) - F(\gamma_t)), & \text{if } \gamma \in [\underline{\gamma}, b'(q_i(\gamma_t))) \end{cases} \tag{63}$$

Thus, condition (ii) is equivalent to $s(\gamma)$ is nondecreasing on $[\underline{\gamma}, b'(q_i(\gamma_t)))$. Hence, graphically, conditions (i'), (ii) and (iii) imply the smallest convex function that lies above S is

$$convS = \begin{cases} S, & \text{if } \gamma \in [\underline{\gamma}, b'(q_i(\gamma_t))), \\ \ell, & \text{if } \gamma \in [b'(q_i(\gamma_t)), \bar{\gamma}], \end{cases}$$

where ℓ is the supporting line of S containing γ_t.

2.4 Linear delegation

In the "linear delegation" case $w(\gamma, q) = \alpha b(q) - d(\gamma)q + C(\gamma)$ (see Kolotilin and Zapechelnyuk, 2019), the conditions are also necessary.

Proposition 2. *If $w_{qq}(q, \theta)/b''(q) = \kappa$ is constant, the conditions in Proposition 1 are also necessary.*

In the linear delegation case, we have

$$s(\gamma) = (\alpha\gamma - d(\gamma))f(\gamma) + \alpha(F(\gamma) - F(\gamma_t)) \tag{64}$$

Corollary 2.1. *Assume $w(\gamma, q) = \alpha b(q) - d(\gamma)q + C(\gamma)$. Then, the price cap allocation is optimal if $s(\gamma)$ is unimodal. In particular,*

- *If $s(\gamma)$ is increasing, no exclusion (and flexible pricing below the price cap) is optimal;*

- *If $s(\gamma)$ is decreasing, a bang-bang allocation where firms either exit or set the price at the price cap is optimal.*

Corollary 2.2. *The bang-bang allocation (where firms either exit or set the price at the price cap) is optimal if and only if there exists $\gamma_t \in [\underline{\gamma}, \bar{\gamma})$ such that $\int_{\gamma_t}^{\gamma} s(\gamma)\,\mathrm{d}\gamma \leq \frac{S(\gamma_t)-S(b'(q_i(\gamma_t)))}{\gamma_t - b'(q_i(\gamma_t))}(\gamma - \gamma_t)$ for all $\gamma \in [\underline{\gamma}, \bar{\gamma})$.*

Corollary 2.3. *Assume $w(\gamma, q) = b(q)/2 + (1/2 - \gamma)q$. Then, the price cap allocation is optimal if $f(\gamma)$ is unimodal. No exclusion is optimal if $f(\gamma)$ is increasing. A bang-bang solution where firms either exit or set the price at the price cap is optimal if $f(\gamma)$ is decreasing.*

Proof sketch. When $w(\gamma, q) = b(q)/2 + (1/2 - \gamma)q$, we have $s(\gamma) = (1-\gamma)f(\gamma)/2 + (F(\gamma) - F(\gamma_t))/2$. Thus, $s'(\gamma) = \gamma f'(\gamma)/2 \overset{sign}{=} f'(\gamma)$, that is, $s(\gamma)$ is increasing (decreasing) if and only if $f(\gamma)$ is increasing (decreasing). □

Example 2.1. Assume $b(q) = q(1-q)$ (so that $\pi(q) = q(1-q) - \gamma q$) and $w(\gamma, q) = \pi(q) + q^2/2$, which is equivalent to the setup in Kolotilin and Zapechelnyuk (2019, Section 4.1). Because $w(\gamma, q) = b(q)/2 + (1/2 - \gamma)q$, Corollary 2.3 applies, and the result is consistent with Kolotilin and Zapechelnyuk (2019, Proposition 1).

2.5 Comparison to Amador and Bagwell (2022)

Because $\gamma_H(\gamma_t) = \max\{b'(q_i(\gamma_t)), \underline{\gamma}\}$, the multiplier $A = G(\gamma_t|\gamma_t) = L(\gamma_H(\gamma_t)|\gamma_t)$ proposed by AB is greater than $\hat{A} = L(b'(q_i(\gamma_t))|\gamma_t)$ that I propose. Thus, condition (i') is a weaker condition than their condition (i). Their condition (ii) is still required to guarantee concavity (see the previous discussions on concavity). Moreover, unlike AB, the two conditions need not hold for all $\gamma_t \in (\underline{\gamma}, \bar{\gamma}]$; instead, they need only hold for the endogenously determined cutoff γ_t. Condition (iii) is absent in AB because they consider the truncated problem at the cutoff.

Consequently, this approach yields weaker sufficient conditions, under which exclusion can be optimal even when there is no fixed cost. Furthermore, in contrast to AB, a bang-bang solution where firms either exit or set the price at the price cap can also be optimal (e.g., when $f(\gamma)$ is decreasing). For example, as shown in Corollary 2.3, if $w(\gamma, q) = b(q)/2 + (1/2 - \gamma)q$, the sufficient condition for the price cap being optimal is a single-peaked density $f(\gamma)$, while AB requires an increasing $f(\gamma)$ (see also Kolotilin and Zapechelnyuk, 2019, Proposition 1).

References

Amador, Manuel and Kyle Bagwell (2013), "The Theory of Optimal Delegation With an Application to Tariff Caps." *Econometrica*, 81, 1541–1599. 10.3982/ECTA9288. [1, 3]

Amador, Manuel and Kyle Bagwell (2022), "Regulating a Monopolist with Uncertain Costs without Transfers." *Theoretical Economics*, 17, 1719–1760. 10.3982/TE4691. [1, 3, 4]

Amador, Manuel, Iván Werning, and George-Marios Angeletos (2006), "Commitment vs. Flexibility." *Econometrica*, 74, 365–396. 10.1111/j.1468-0262.2006.00666.x. [1]

Bryson, Arthur Earl and Yu-Chi Ho (1975), *Applied Optimal Control: Optimization, Estimation, and Control*, New York, NY: Taylor & Francis. [3, 7]

Clarke, Francis (2013), *Functional Analysis, Calculus of Variations and Optimal Control*, 264 of Graduate Texts in Mathematics, London: Springer. 10.1007/978-1-4471-4820-3. [3, 7]

Guo, Yingni (2016), "Dynamic Delegation of Experimentation." *American Economic Review*, 106, 1969–2008. 10.1257/aer.20141215. [1]

Hellwig, Martin F. (2010), "Incentive Problems With Unidimensional Hidden Characteristics: A Unified Approach." *Econometrica*, 78, 1201–1237. 10.3982/ECTA7726. [3, 7]

Kamien, Morton I. and Nancy L. Schwartz (1971), "Sufficient Conditions in Optimal Control Theory." *Journal of Economic Theory*, 3, 207–214. 10.1016/0022-0531(71)90018-4. [3]

Kolotilin, Anton and Andriy Zapechelnyuk (2019), "Persuasion Meets Delegation." February, arXiv: 1902.02628. 10.48550/arXiv.1902.02628. [10, 11]