Incentivizing Agents through Ratings*

Peiran Xiao†

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Abstract

I study the optimal design of ratings to motivate agent investment in quality when transfers are unavailable. The principal designs a rating scheme that maps the agent's quality to a (possibly stochastic) score. The agent has private information about his ability, which determines his cost of investment, and chooses the quality level. The market observes the score and offers a wage equal to the agent's expected quality. For example, a school incentivizes learning through a grading policy that discloses the student's quality to the job market.

When restricted to deterministic ratings, I provide necessary and sufficient conditions for the optimality of simple pass/fail tests and lower censorship. In particular, when the principal's objective is expected quality, pass/fail tests are optimal if the agent's ability distribution is concentrated towards the top, while lower censorship is optimal if the ability distribution is concentrated towards the mode. The results also generalize existing results in optimal delegation with an outside option, as pass/fail tests (lower censorship) correspond to take-it-or-leave-it offers (threshold delegation). Additionally, I provide sufficient conditions under which stochastic ratings outperform deterministic ratings and under which they do not.

Keywords: Rating design, moral hazard, pass/fail tests, delegation, optimal control.

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†Department of Economics, Boston University. Email: pxiao@bu.edu.

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1 Introduction

In many economic applications, a principal seeks to motivate agents' performance or investment in quality, but monetary transfers between them are prohibited or limited. In these situations, the principal can instead incentivize agents through a rating scheme (or disclosure policy) that reveals information about their endogenous quality to the market. When the market rewards agents based on this information, ratings can provide reputational incentives for agents.

For example, consider a school in which students make productive investments to improve their quality (i.e., human capital). Suppose the school wants to incentivize student investment to achieve better placement outcomes, maximize tuition fees, or encourage human capital formation. To maximize its objective, the school designs a grading rule that discloses information about students' endogenous quality to the job market. Similarly, regulatory certifiers who care about consumer welfare can motivate firm investment in product quality through quality certification that reveals information about the product quality to consumers.[1] Employers (e.g., pre-doc positions) may pay a fixed wage to employees and induce effort through ratings that provide information about their performance and abilities to future employers. In these examples, the market pays the agent the expected value of his endogenous quality (or inherent ability) conditional on the rating result. By contrast, transfers contingent on the quality or the rating between the principal and agent are often infeasible in practice or prohibited by law.

Various rating schemes are used in these environments to motivate agents. A frequently observed scheme is *pass/fail* tests. Licensing exams, such as bar examinations, are often pass/fail. Pass/fail is also ubiquitous in product certification, such as UL Certifications and ISO Certifications. Another prevalent scheme is *lower censorship*, which reveals quality if and only if it exceeds a threshold or minimum standard. For example, some schools release precise scores above a failing grade. In product certification, lower censorship takes the form of quality assurance, which censors low-quality products that do not meet the standard and prevents them from being sold on the market. Yet another form is coarse letter grades or star ratings that have multiple thresholds. For instance, students who meet the lower standard but not the higher one get a "low-pass" grade. Alternatively, ratings may involve randomness, such as random inspection or disclosure of product quality. For example, the certifier may use an algorithm that determines the probability of checking or disclosing the product quality.

[1] Regulatory or NGO certifiers care about overall product quality because of consumer welfare (see, e.g., Zapechelnyuk, 2020; Bizzotto and Harstad, 2023; Vatter, 2023) or spillovers of quality. Examples include restaurant hygiene ratings, Medicare Star Ratings, and certifications for energy efficiency or product safety.

In this paper, I study the optimal design of rating schemes to motivate agent investment in quality when transfers are unavailable. Instead, the principal designs a rating scheme (à la Blackwell) that maps the agent's quality to a (possibly stochastic) score. The agent has private information about his ability, which determines his cost of investment, and chooses the quality level. The market observes the score and offers a wage equal to the agent's expected quality.

At first glance, full revelation (or full disclosure) of quality might seem to be the optimal scheme because any marginal investment in quality will be revealed to the market. This is true for a utilitarian principal who has the same preference as the agent. However, when the principal wants to incentivize higher investment in quality, a minimum standard can provide stronger incentives for some agents, as they need to invest more in quality to separate themselves from the low levels that fail to meet the standard. Therefore, tests with minimum standards (or multiple thresholds), such as pass/fail and coarse grading, can be optimal.[2] Alternatively, stochastic rating schemes can potentially provide stronger incentives for some types than deterministic rating schemes.

To characterize the optimal rating scheme, I reduce the rating design problem to the equivalent problem of designing an incentive-compatible direct mechanism that consists of a quality function and an *interim* wage function. The interim wage function maps the agent's type to the expected wage he receives from the market in equilibrium. Unlike standard principal-agent models, the agent's wage is offered by the market equal to his expected quality conditional on the score and thus must be induced by a rating scheme. Therefore, the mechanism design problem is subject to a feasibility constraint that the interim wage is a mean-preserving spread of the quality in the quantile space.

My first set of results concerns the optimal *deterministic* rating schemes. A deterministic rating scheme either fully reveals quality or pools some qualities to the same score. In the latter case, among the qualities that are pooled to the same score, only the lowest one will be chosen by the agent.[3] Thus, the interim wage always equals the quality, as the market can perfectly infer the agent's quality from his score. Using optimal control methods, I provide sufficient conditions for the optimal deterministic rating scheme to be lower censorship or a simple pass/fail test. The conditions are also necessary if the principal's marginal payoff from the agent's quality is linear in (a transformation of) the quality. In particular, when the principal maximizes expected quality, lower censorship is

[2]This argument does not hinge on cognitive or technological costs (or constraints) of precise information, which are not considered in this paper. These costs or constraints will make pass/fail tests and coarse grading more likely to be optimal.

[3]This argument hinges on the assumption that quality can be chosen deterministically and no longer holds if agent investment determines quality stochastically.

optimal if and only if the agent's ability distribution is concentrated around the mode (e.g., unimodal density). A pass/fail test maximizes the expected quality if and only if the ability distribution is concentrated towards the top (e.g., increasing density). Otherwise, if the ability distribution is concentrated towards the bottom (e.g., decreasing density), lower censorship with a minimum standard that every type will meet in equilibrium maximizes the expected quality. Intuitively, when there are more high types, it is more profitable to set a high minimum standard to induce higher investment in quality from high types, even if it excludes some low types. Specifically, the optimal minimum standard is such that passing requires even the highest type to invest more than he would under full revelation. On the other hand, when there are more low types, excluding them to incentivize high types becomes unprofitable, so the optimal minimum standard will allow the lowest type to barely reach it in the equilibrium.

I focus on pass/fail tests and lower censorship because they are the most prevalent rating schemes in practice. They are also simple as they contain one (or fewer) threshold or minimum standard. In addition to them, I characterize the optimal deterministic ratings for general type distributions and principal preferences in Appendix B, which may contain multiple thresholds. For example, if the ability density is bimodal, the quality-maximizing deterministic rating can take the form of high-pass/low-pass/fail.

My results also have implications for optimal delegation because the deterministic rating design problem is equivalent to optimal deterministic delegation with an outside option (see also Zapechelnyuk, 2020). In the delegation problem (à la Holmstrom (1984)), the principal determines a set of permissible actions and delegates the agent to choose one from the set (or the outside option). Similarly, in the deterministic rating design problem, the principal effectively designs a set of qualities for the agent to choose.[4] Thus, pass/fail tests correspond to take-it-or-leave-it offers, while lower censorship corresponds to threshold delegation. Analogously, other deterministic rating schemes also have counterparts in delegation.

My second set of results considers settings where stochastic rating schemes are allowed. A natural question is whether the principal can benefit from introducing randomness to the rating scheme. To answer this question, I first provide sufficient conditions under pass/fail tests and full revelation remain optimal. In the quality maximization case, pass/fail tests remain optimal if the ability density is increasing. Second, I identify conditions under which stochastic ratings strictly improve on deterministic ratings. For example, a noisy test that partially pools low quality with high quality enables the princi-

[4]To see this, when multiple qualities are pooled to the same score, the lowest quality among them will strictly dominate others.

pal to increase the incentives for low types at the cost of incentives for high types, which can potentially increase the overall expected quality. This is true when the ability density has a heavy tail—that is, there are a few very high-ability agents.

As an extension, I consider the ability signaling case where the market values the agent's exogenous ability instead of endogenous quality. In other words, the agent's effort is signaling rather than productive. The rating design problem can also be reduced to a mechanism design problem subject to a feasibility constraint but one where the interim wage must be a mean-preserving spread of the *ability* in the quantile space. If the agent's cost is linear in quality, the quality-maximizing rating is always deterministic and induces full separation if and only if the ability distribution is regular in the sense of Myerson (1981).

Methodologically, the paper uses recent advances in optimal control methods to address possible jumps in the optimal quality scheme (Hellwig, 2008, 2010; Clarke, 2013). Because there are no transfers, the Myersonian approach is not applicable. Neither is the standard optimal control method (e.g., Guesnerie and Laffont (1984)) because they require the quality scheme (i.e., state variable) to be absolutely continuous.[5] Thus, I use the maximum principle formulated by Hellwig (2008, 2010) to handle the monotonicity constraint on the quality scheme without assuming its absolute continuity. Moreover, because of the outside option, the optimal quality scheme can have a jump at the cutoff type. I use the switching condition in the hybrid maximum principles (Clarke, 2013; Bryson and Ho, 1975) to characterize the optimal cutoff.

The paper makes three contributions to the literature. First, I provide a unified framework to study the optimal rating scheme to motivate agents, with a focus on simple pass/fail tests. In this general framework, the principal can have a state-dependent preference and design stochastic rating schemes, while the agent's effort can be either productive or signaling.

Second, my results for optimal deterministic ratings generalize existing results in optimal delegation with an outside option (e.g., Amador and Bagwell (2022); Kartik, Kleiner, and Van Weelden (2021); see details in the literature review). I provide necessary and sufficient conditions for the optimality of threshold delegation and take-it-or-leave-it offers, allowing for general preferences of the principal that may depend on the agent's type (i.e., state-dependent) or nonlinear in the agent's action (i.e., nonlinear delegation). In particular, take-it-or-leave-it offers and bang-bang allocations (i.e., binary actions in

[5]With type-contingent transfers, the optimal scheme has no jumps (see, e.g., Mussa and Rosen (1978) and Kamien and Schwartz (2012, Section 18)), so one can assume absolute continuity and use its derivative as a control variable.

equilibrium) remain underexplored in this literature.[6] Additionally, through the equivalence established by Kolotilin and Zapechelnyuk (2025) between delegation problems and Bayesian persuasion problems, the results contribute to the persuasion literature, especially in the nonlinear case.

Third, to my knowledge, this is the first paper that allows for *stochastic* ratings in optimal rating design to motivate agent investment in quality without transfers. Even in the case where the principal maximizes expected quality, the literature has focused on optimal deterministic ratings (e.g., Zapechelnyuk (2020); Rodina and Farragut (2020); Rayo (2013); Zubrickas (2015)). By contrast, I explore stochastic ratings using the *interim* approach by Saeedi and Shourideh (2020) that reduces the rating design problem to the optimization over interim wage functions rather than Blackwell experiments themselves (see also Doval and Smolin (2022)).

Literature Review. This paper incorporates two strands of literature on the optimal rating design to motivate agents when the market rewards them with the expected value. A strand of literature assumes the market values the agent's *endogenous* quality or effort (Albano and Lizzeri (2001); Saeedi and Shourideh (2020, 2023); Zapechelnyuk (2020); Rodina and Farragut (2020); Boleslavsky and Kim (2021); Vatter (2023)). Zapechelnyuk (2020) studies the optimal *deterministic* quality certification to incentivize sellers' investment in product quality and characterize sufficient conditions for lower censorship and pass/fail certifications, and the conditions for pass/fail require small variations in agents' abilities.[7] Compared to the literature, my conditions for lower censorship and especially pass/fail tests are less restrictive. I also allow for state-dependent preferences and stochastic rating schemes.[8]

Another strand of literature assumes the market values the *exogenous* abilities à la Spence's (1973) signaling model (Dewatripont et al. (1999); Rayo (2013); Zubrickas (2015); Rodina (2020); Hörner and Lambert (2021); Onuchic and Ray (2023); Camboni et al. (2024)). Rayo (2013) and Zubrickas (2015) characterize the conditions under which the effort-maximizing deterministic rating scheme induces full separation or pooling of agents. In Appendix C, I provide necessary and sufficient conditions for full separation to be optimal while allowing for stochastic ratings and general objective functions.

[6]The delegation literature has focused on the case without the outside option until recently. The papers that have the outside option either rule out bang-bang allocations (Amador and Bagwell (2022)) or assume state-dependent principal preferences (Kartik et al. (2021)).

[7]Rodina and Farragut (2020) also characterize the properties of the effort-maximizing deterministic grading rules when the distribution is sufficiently concave, convex, and single-peaked.

[8]Boleslavsky and Kim (2021) consider stochastic rating schemes without transfers but assume agent investment improves the *distribution* of his quality.

This results on optimal deterministic ratings also contribute to the literature on optimal delegation with outside option. Amador and Bagwell (2022) study the problem of regulating a monopolist without transfers and characterize sufficient conditions for threshold delegation (i.e., price caps) to be optimal. Compared to them, my conditions for threshold delegation are *necessary* and sufficient, thereby allowing for the optimality of a bang-bang allocation where the firm either shuts down or always sets the price at the cap. This bang-bang allocation, which can also be implemented by a take-it-or-leave-it offer, is more realistic because monopolists rarely set prices below the cap.[9] Kartik et al. (2021) study delegation in veto-bargaining with an outside option when the principal has a state-independent single-peaked preference. They identify the necessary and sufficient conditions for the optimality of interval (and full) delegation and take-it-or-leave-it offers among possibly stochastic delegation mechanisms. By contrast, I allow for state-*dependent* preferences and stochastic rating schemes. Saran (2022) studies optimal delegation with outside option using a dynamic optimization approach and identifies sufficient conditions for the optimal mechanism to have at most finitely many discontinuities.

The method I use in characterizing optimal deterministic ratings develops the Lagrangian methods in the delegation literature advanced by Amador, Werning, and Angeletos (2006) (see also Amador and Bagwell (2013, 2022)) to address jumps in the optimal allocation (due to the outside option) using optimal control methods (Bryson and Ho (1975); Hellwig (2008, 2010); Clarke (2013)). Under deterministic ratings, this method tackles the (deterministic) delegation problem directly without invoking the equivalence to persuasion and allows for nonlinear delegation. Moreover, it extends to stochastic ratings using the interim wage function.[10]

2 The Model

2.1 Setup

The model has three players: a principal, an agent, and a market. The agent has a private type θ, which has a continuous distribution $F(\theta)$ with full support $\Theta = [\underline{\theta}, \bar{\theta}]$, where $0 \leq \underline{\theta} < \bar{\theta}$, and continuous density $f(\theta)$. He can choose a quality level $q \in Q = [0, q_{\max}]$ at cost $c(q, \theta) = c(q)/\theta$, which is strictly increasing and convex in q. Assume that $c(0) =$

[9]Under their sufficient conditions, the bang-bang allocation is never optimal. See also Halac and Yared (2022) for the optimality of bang-bang incentive schemes.

[10]Despite the equivalence between deterministic rating schemes and deterministic delegation, stochastic rating schemes (in Section 5) are *not* equivalent to stochastic delegation.

$c'(0) = 0$. Assume without loss that $q_{\max} = \{q > 0 \colon c(q)/\bar{\theta} = q\}$, which uniquely exists by the convexity of $c(q)$.

The principal has a utility function given by $v(q, \theta)$, which is twice continuously differentiable and satisfies $v_{qq}(q, \theta) \leq 0$, $v(0, \theta) = 0$, and $v_q(0, \theta) > 0$ for all $q \in Q$ and $\theta \in \Theta$. The principal does not observe θ, and it does not matter whether the principal observes q as long as the rating takes it as input. If she observes q, the rating scheme is a disclosure policy that garbles the quality; otherwise, it is a test that inputs the quality and outputs a score. Assume there are no transfers between the principal and agent. Instead, the principal can design a rating scheme (i.e., Blackwell experiment) $\pi \colon Q \to \Delta S$, which is publicly observed, to reveal information about the agent's quality q (and hence type θ) to the market and provide reputational incentives. The agent can choose whether to participate in the rating scheme (i.e., take the test). If he takes the test, the market observes a signal $s \sim \pi(q)$. Otherwise, the market observes a null signal $s = \varnothing$.

The market values the agent's quality q. Assume the market is competitive and has a payoff $-(\omega - q)^2$ when she pays a wage ω to an agent of quality q. After observing the score s, the market updates her belief of the agent's quality to $\mu_s \in \Delta Q$ using Bayes' rule, and then offers him a wage equal to the expected value $\omega(s) = \mathbf{E}[q|s] \equiv \mathbf{E}_{\mu_s}[q]$. Thus, if the agent takes the test, his interim wage, as a function of his quality q, is $\hat{w}(q) = \mathbf{E}_{s \sim \pi(q)}[\omega(s)]$, and he chooses $q \in Q$ to maximize his payoff $\hat{w}(q) - c(q)/\theta$. For convenience, I scale the payoff by θ and define $u(q, \theta) = \theta \hat{w}(q) - c(q)$. If the agent chooses not to take the test, the market offers him a wage $\omega(\varnothing)$ based on the null signal.

Timing. First, the agent privately learns his type $\theta \in \Theta$. Then, the principal commits to a rating scheme $\pi \colon Q \to \Delta S$. Next, the agent chooses a quality level $q \in Q$ and whether to take the test. Finally, the market observes the score s and offers a wage $\omega(s) = \mathbf{E}[q|s]$.

Solution Concept. I use weak Perfect Bayesian Equilibrium as the solution concept. In any equilibrium, if an agent does not take the test, he must choose $q = 0$ because investment is costly. Thus, the market must believe that he has chosen $q = 0$ and offer $\omega(\varnothing) = 0$ accordingly. Hence, the problem does not suffer from the issue of multiple equilibrium outcomes, in contrast to signaling games. In particular, the agent's payoff from choosing the outside option (not taking the test) is zero in any equilibrium, as stated in the following lemma.

Lemma 1. *In any equilibrium, if an agent does not take the test on the equilibrium path, then he chooses $q = 0$, and the market will offer him $\omega(\varnothing) = 0$.*

Downward Bias. Define the agent's quality choice under full revelation as

$$q_f(\theta) = \arg\max_{q \in Q} \theta q - c(q) = c'^{-1}(\theta).$$

Define the principal-optimal quality scheme as $q_e(\theta) = \arg\max_{q \in Q} v(q, \theta)$.

Assumption 1 (Downward bias)**.** $q_f(\theta) \leq q_e(\theta)$ for all $\theta \in [\underline{\theta}, \bar{\theta}]$.

Because $v_{qq}(q, \theta) \leq 0$, this assumption is equivalent to $v_q(q_f(\theta), \theta) \geq 0$ for all $\theta \in [\underline{\theta}, \bar{\theta}]$. In other words, the principal wants to incentivize the agent to choose a higher q.[11]

2.2 Discussion of Assumptions

The market values quality. I assume the market values the (endogenous) quality q rather than the (exogenous) ability θ to shut down signaling.[12] This captures the cases in (i) the school example when learning is productive rather than signaling and (ii) the product certification example when the consumer values the product quality. In Appendix C, I assume the market values the ability θ à la Spence's (1973) signaling model.

The misalignment of incentives. I assume the principal wants to incentivize higher investment in quality than the agent. For example, the principal internalizes only partially the agent's cost. Below, I provide several strands of examples.

First, the profit-maximizing principal may not care about the costs. For example, an employer only wants to induce higher outputs q from employees.

Second, the principal may want to induce higher quality investment because of social spillovers (e.g., Zubrickas (2015)). Similarly, the regulatory certifier maximizes a weighted sum of the average quality and the firms' profit and (Bizzotto and Harstad, 2023).[13]

Third, this misalignment can result from more complicated models. For example, the school maximizes students' placement outcomes (i.e., expected wage), which is equal to the expected quality, for reputation or alumni donation.[14]

[11] Alternatively, if the principal wants to induce *lower* investments than the agent, she will use a noisy rating—i.e., a garbling of the fully revealing test such that $\hat{w}'(q) < 1$.

[12] The market value can be easily generalized to a function of q if the cost function is adjusted accordingly.

[13] To see this, $\mathbf{E}[\alpha q(\theta) + (1-\alpha)U(\theta)] = \mathbf{E}[\alpha q(\theta) + (1-\alpha)(w(\theta) - c(q(\theta), \theta))] = \mathbf{E}[q(\theta) - \alpha c(q(\theta), \theta)]$.

[14] Other examples include Onuchic and Ray (2023) and Zapechelnyuk (2020). In Onuchic and Ray (2023, Section 4), the school maximizes the expected tuition fee equal to $\mathbf{E}[q(\theta) - \alpha c(q(\theta), \theta)]$. In Zapechelnyuk (2020), the regulatory certifier maximizes consumer surplus, which equals to the expected quality.

The role of (no) transfers. I rule out transfers to focus on the role of ratings in providing incentives. With transfers contingent on the rating result (or the agent's quality), the design of ratings becomes irrelevant because contingent transfers can provide incentives in place of $w(\theta)$ (see Appendix E.1). I also discuss a *constant* testing fee in Appendix E.

Commitment to the rating scheme. I assume the principal can commit to the rating scheme. This assumption is innocuous because the principal has no incentives to tamper with the ratings, as her objective $v(q,\theta)$ does not depend on the wage offered by the market, and there are no transfers contingent on the rating results.[15]

3 Revelation Principle and Feasibility

Consider a direct mechanism $(q(\theta), s(\theta))$. If the agent accepts this mechanism, he reports his type θ, and is then required to choose quality level $q(\theta)$ and receives a (possibly stochastic) score $s(\theta)$ drawn from $\pi(q(\theta))$. The rating scheme $\pi\colon Q \to \Delta S$ is an implementation of this direct mechanism, which does not require the agent's quality q to be observable by the principal, as long as it is taken as input by the rating scheme. By the revelation and taxation principles, these two mechanisms are equivalent—choosing q is equivalent to reporting θ.

Formally, say a quality function $q\colon \Theta \to Q$ is *implementable* by a rating scheme $\pi\colon Q \to \Delta S$ if the induced interim wage, $\hat{w}(q) = \mathbf{E}_{s\sim\pi(q)}[\omega(s)]$, satisfies the incentive compatibility constraint

$$\theta\hat{w}(q(\theta)) - c(q(\theta)) \geq \theta\hat{w}(q') - c(q') \quad \text{for all } \theta \in \Theta \text{ and } q' \in Q. \tag{1}$$

Instead of optimizing over Blackwell experiments, it is easier to work with the interim wage $\hat{w}\colon Q \to \mathbb{R}_+$ induced by π. Therefore, I focus on a direct mechanism $(q(\theta), w(\theta))$ consisting of the quality function $q(\theta)$ and the interim wage function $w(\theta) = \hat{w}(q(\theta))$. Unlike a standard transfer between the principal and agent, the interim wage $w(\theta)$ is offered by the market equal to the agent's expected quality conditional on the score. Thus, the interim wage must be induced by a rating scheme, as captured by the following definition of feasibility.

[15] Alternatively, when the principal's objective is the expected wage, which equals the expected quality, she *does* have incentives to manipulate the rating results. However, the rating scheme is still *credible* in the sense of Lin and Liu (2024), as the principal cannot profit from tampering with the rating scores while keeping the score distribution unchanged.

Definition 1. A direct mechanism $(q(\theta), w(\theta))$ is *feasible* if there exists a rating scheme $\pi\colon Q \to \Delta S$ such that $w(\theta) = \hat{w}(q(\theta)) \equiv \mathbf{E}_{s\sim\pi(q(\theta))}[\mathbf{E}[\tilde{q}|s]]$.

Say a quality function $q\colon \Theta \to Q$ is *implementable* by a direct mechanism $(q(\theta), w(\theta))$ if it satisfies the incentive compatibility constraint

$$\theta w(\theta) - c(q(\theta)) \geq \theta w(\theta') - c(q(\theta')) \quad \text{for all } \theta, \theta' \in \Theta. \tag{2}$$

The following lemma establishes the equivalence between the direct mechanism and the rating mechanism, thereby allowing one to focus on feasible direct mechanisms $(q(\theta), w(\theta))$.

Lemma 2. *An allocation $q(\theta)$ is implementable by the rating scheme $\pi(q)$ if and only if it is implementable by a feasible direct mechanism $(q(\theta), w(\theta))$.*

Remark 1. According to this lemma, eliciting the agent's information through a menu of tests has no value because any implementable direct mechanism $(q(\theta), w(\theta))$ can be implemented by a single test.

By the standard argument, incentive compatibility of a direct mechanism $(q(\theta), w(\theta))$ is equivalent to the monotonicity of $w(\theta)$ (and $q(\theta)$) and the envelope condition (see Lemma A.1)

$$\theta w(\theta) - c(q(\theta)) = \int_{\underline{\theta}}^{\theta} w(x)\,\mathrm{d}x + \underline{U}, \tag{3}$$

where $\underline{U} = \underline{\theta} w(\underline{\theta}) - c(q(\underline{\theta}))$. In addition to incentive compatibility, $(q(\theta), w(\theta))$ must be feasible (Definition 1) in the sense that $w(\theta)$ must be induced by a rating scheme.

4 Optimal Deterministic Ratings

4.1 Principal's Problem

In this section, I restrict attention to *deterministic* rating schemes $\pi\colon Q \to S$, which either fully reveal the quality or pool multiple qualities into a single score. It is without loss to restrict attention to right-continuous $\pi\colon Q \to S$, as rating schemes that are not right-continuous cannot implement any quality scheme $q(\theta)$.[16] When quality is fully revealed, the market learns the quality. When multiple qualities are mapped to the same score s, the lowest quality $\min\{q : \pi(q) = s\}$ (which exists by right-continuity) strictly

[16] For example, $\pi(q) = \mathbf{1}[q > 1]$ cannot implement any quality scheme because the agent will choose $q > 1$ as close to 1 as possible.

dominates all other $q \in \{q : \pi(q) = s\}$, so only the lowest quality will be chosen, and the market also learns the quality (see also Zapechelnyuk, 2020, Claim 1). Therefore, in either case, the interim wage is $w(\theta) = q(\theta)$.

Lemma 3. *Under deterministic ratings, the interim wage function is* $w(\theta) = q(\theta)$.

By the revelation principle and Lemma 2, looking for the optimal deterministic rating scheme π is equivalent to looking for the optimal quality scheme $q(\theta)$. Thus, I shall focus on the quality scheme and be casual in distinguishing the two.

Now the principal's problem becomes

$$\text{[P]} \quad \max_{q(\theta)} \int_{\underline{\theta}}^{\bar{\theta}} v(q(\theta), \theta) \, \mathrm{d}F(\theta) \tag{4}$$

subject to, for all $\theta \in [\underline{\theta}, \bar{\theta}]$,

$$\theta q(\theta) - c(q(\theta)) \geq 0 \tag{IR}$$

$$q(\theta) \text{ increasing} \tag{IC-Mon}$$

$$\theta q(\theta) - c(q(\theta)) = \int_{\underline{\theta}}^{\theta} q(x) \, \mathrm{d}x + \underline{U} \tag{IC-Env}$$

where $\underline{U} = \theta q(\underline{\theta}) - c(q(\underline{\theta}))$.

The principal's problem [P] is equivalent to delegation (Holmstrom (1984)) with an *outside option*, where the principal determines a set of permissible qualities q and delegates the agent to choose one from the set or the outside option $q = 0$ (and not taking the test) (see also Amador and Bagwell (2022)). Indeed, by fully revealing quality, the principal imposes no restrictions on the delegation set. By pooling multiple qualities to the same score, the principal effectively removes all but the lowest of these qualities from the delegation set. Specifically, a quality scheme $q(\theta)$ is equivalent to a delegation set $\{q(\theta) : \theta \in \Theta\}$.

An incentive-compatible quality scheme $q(\theta)$ consists of pooling and full revealing intervals and contains at most countably many jump discontinuities.[17] In particular, the outside option can lead to a jump at the cutoff type θ_0, who is indifferent between choosing the outside option ($q = 0$) and a positive quality $q_i(\theta)$ given by

$$\theta q_i(\theta) - c(q_i(\theta)) = 0 \text{ and } q_i(\theta) > 0.\text{[18]}$$

[17] See Melumad and Shibano (1991, Proposition 1) and Alonso and Matouschek (2008, Lemma 2). The proof that allows for general preferences is in Appendix F

[18] By the convexity of $c(q)$ and $c(0) = 0$, a unique $q_i(\theta) \geq q_f(\theta)$ exists for all $\theta \in [\underline{\theta}, \bar{\theta}]$. In particular,

Lemma 4. *There exists a cutoff type* $\theta_0 \in [\underline{\theta}, \bar{\theta}]$ *such that* $q(\theta) = 0$ *for all* $\theta \in [\underline{\theta}, \theta_0)$ *and* $q(\theta) > 0$ *for all* $\theta \in (\theta_0, \bar{\theta}]$. *If* $\theta_0 \in (\underline{\theta}, \bar{\theta})$, *then* $q(\theta_0) = q_i(\theta_0)$.

4.2 Lower Censorship and Pass/fail Tests

In this paper, I will focus on two classes of deterministic ratings with minimum standard—lower censorship and pass/fail tests.

Definition 2. *Lower censorship* is a deterministic rating $\pi\colon Q \to Q \cup \{\text{fail}\}$ that reveals the quality q if $q \geq q_0$ for some $q_0 \in Q$ and gives a "fail" otherwise, i.e.,

$$\pi(q) = \begin{cases} q, & \text{if } q \geq q_0, \\ \text{fail}, & \text{otherwise}. \end{cases}$$

Definition 3. A *pass/fail test* is a deterministic rating $\pi\colon Q \to \{\text{pass}, \text{fail}\}$ that gives a "pass" if $q \geq q_0$ for some $q_0 \in Q$ and a "fail" otherwise, i.e.,

$$\pi(q) = \begin{cases} \text{pass}, & \text{if } q \geq q_0, \\ \text{fail}, & \text{otherwise}. \end{cases}$$

The threshold q_0 in these definitions is called a *minimum standard*. A *fully revealing test* is a special case of lower censorship where the minimum standard $q_0 = 0$. The minimum standard $q_0 \in Q$ leads to a cutoff type $\theta_0 = c(q_0)/q_0 \in [0, \bar{\theta}]$ (such that $q_i(\theta_0) = q_0$).

Define $\theta_c\colon [\underline{\theta}, \bar{\theta}] \to [\underline{\theta}, +\infty)$ as $\theta_c(\theta) = c'(q_i(\theta))$. Recall that $q_f(\theta) = c'^{-1}(\theta)$, so $\theta_c(\theta)$ is the type that would choose $q = q_i(\theta)$ under full revelation—i.e., $q_i(\theta) = q_f(\theta_c(\theta))$. For example, if $c(q) = q^2/2$, then $\theta_c(\theta) = q_i(\theta) = 2\theta$.

Lower censorship with minimum standard q_0 induces a quality scheme that potentially consists of (i) an *exclusion* region $[\underline{\theta}, \theta_0)$ where agents choose $q = 0$, (ii) a *bunching* region $[\theta_0, \theta_c(\theta_0))$ where agents are bunched at the threshold $q_0 = q_i(\theta_0)$, and (iii) a *fully revealing* region $[\theta_c(\theta_0), \bar{\theta}]$ where agents choose $q_f(\theta)$.[19]

$$q(\theta) = \begin{cases} 0, & \text{if } \theta \in [\underline{\theta}, \theta_0) \\ q_i(\theta_0), & \text{if } \theta \in [\theta_0, \theta_c(\theta_0)) \\ q_f(\theta), & \text{if } \theta \in [\theta_c(\theta_0), \bar{\theta}] \end{cases} \quad (5)$$

$q_i(\bar{\theta}) = q_{\max}$. Define $q_i(0) = 0$ (because $\lim_{q \to 0} c(q)/q = 0$).

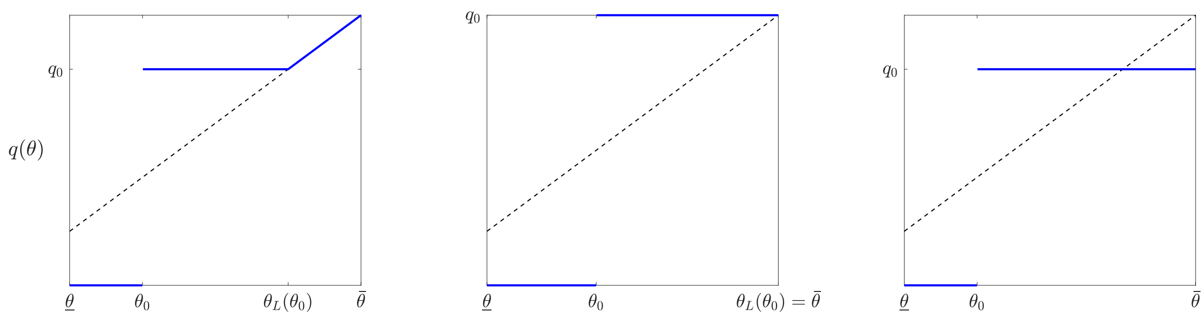
[19]By convention, $[x, y)$, (x, y), and $[x, y]$ represent the empty set if $x \geq y$. Some of these regions can be empty if (i) $\theta_0 \leq \underline{\theta}$, (ii) $\theta_c(\theta_0) \leq \underline{\theta}$, or (iii) $\theta_c(\theta_0) \geq \bar{\theta}$.

It is useful to define the start of fully revealing region by

$$\theta_1(\theta_0) = \max\{\min\{\theta_c(\theta_0), \bar{\theta}\}, \underline{\theta}\}.$$

Analogously, a pass/fail test with minimum standard $q_0 \in Q$ induces the quality scheme that potentially consists of the exclusion region and the bunching region

$$q(\theta) = \begin{cases} 0, & \text{if } \theta \in [\underline{\theta}, \theta_0), \\ q_i(\theta_0), & \text{if } \theta \in [\theta_0, \bar{\theta}]. \end{cases}$$



Note: $q(\theta)$ on the left can be induced by lower censorship with minimum standard q_0; $q(\theta)$ on the right can be induced by a pass/fail test with minimum standard q_0; $q(\theta)$ in the center can be induced by both.

Figure 1: $q(\theta)$ induced by lower censorship and pass/fail

There are two caveats. First, it is possible that $c(q_0)/q_0 < \underline{\theta}$. In this case, $\theta_0 = c(q_0)/q_0$ is a hypothetical cutoff "type" below $\underline{\theta}$, and the exclusion region $[\underline{\theta}, \theta_0)$ is empty.

Second, for lower censorship with minimum standard $q_0 > q_f(\bar{\theta})$, we have $\theta_c(\theta_0) > \bar{\theta}$, so the fully revealing region $[\theta_c(\theta_0), \bar{\theta}]$ is empty. In words, the minimum standard is so high that no one will choose any strictly higher quality in equilibrium. Thus, the lower censorship induces the same quality scheme as a pass/fail test with the same minimum standard q_0.[20]

4.3 Linear Delegation

In this subsection, I focus on objective functions in the form of $v(q, \theta) = \beta(\theta)q - \alpha c(q) + d(\theta)$ for some functions $\beta, d \colon \Theta \to \mathbb{R}$ and constant $\alpha \geq 0$. Thus, the "relative concavity" of the principal's and the agent's preferences, given by $-v_{qq}(q, \theta)/c''(q) = \alpha$, is constant. This

[20] Note that a pass/fail test is *not* a special case of lower censorship because the off-path strategies $q > q_0$ lead to different outcomes, although they induce the same quality scheme in equilibrium.

case is referred to as "linear delegation" in Kolotilin and Zapechelnyuk (2025) because the principal's marginal payoff from the agent's action q is linear in (a transformation of) q.

Condition (LD). The principal's objective function is $v(q,\theta) = \beta(\theta)q - \alpha c(q) + d(\theta)$ for some functions $\beta, d\colon \Theta \to \mathbb{R}$ and constant $\alpha \geq 0$ such that $\beta(\theta) \geq \alpha\theta$ (by Assumption 1).

Quality maximization (i.e., $v(q,\theta) = q$) is a simple case that satisfies Condition LD, which will be used as a running example throughout this subsection. Section 4.3.3 provides more examples of linear delegation, including quadratic loss utilities.

4.3.1 Necessary and Sufficient Conditions

Define the characteristic functions $r\colon \mathbb{R}_+ \to \mathbb{R}$ and $R\colon \mathbb{R}_+ \to \mathbb{R}_+$, which generalize the density $f(\theta)$ and the distribution $F(\theta)$ by incorporating the principal and agent's preferences, as

$$r(\theta) = (\beta(\theta) - \alpha\theta)f(\theta) - \alpha(F(\theta) - F(\theta_0)) \text{ for all } \theta \geq 0, \tag{6}$$

$$R(\theta) = \int_{\underline{\theta}}^{\theta} r(x)\,\mathrm{d}x = \int_{\underline{\theta}}^{\theta} \beta(x)f(x)\,\mathrm{d}x - \alpha\theta(F(\theta) - F(\theta_0)). \tag{7}$$

Example (Quality Maximization)**.** If $v(q,\theta) = q$, then $r(\theta) = f(\theta)$ and $R(\theta) = F(\theta)$.

The characterization function $r(\theta)$ is determined by the density $f(\theta)$, objective $v(q,\theta)$, the cost function $c(q)$, which can be viewed as a generalization of the density function. Note that $r(\theta)$ is defined on $\mathbb{R}_+$, which requires extending $F(\theta)$ and $f(\theta)$ from $[\underline{\theta}, \bar{\theta}]$ to $\mathbb{R}_+$. By convention, $f(\theta) = 0$ for all $\theta < \underline{\theta}$ and $F(\theta) = 0$ for all $\theta < \underline{\theta}$; $f(\theta) = 0$ for all $\theta > \bar{\theta}$ and $F(\theta) = 1$ for all $\theta > \bar{\theta}$.

Observation 1. *The function $R(\theta)$ is continuous and satisfies the following properties:*

(i) $R(\theta)$ is increasing on $[0,\theta_0]$ and decreasing on $[\bar{\theta}, +\infty)$.

(ii) R can be non-differentiable and have a convex kink at $\underline{\theta}$ and a concave kink at $\bar{\theta}$ (due to possible discontinuities of f).

Define

$$A(\theta_0) = \frac{R(\theta_c(\theta_0)) - R(\theta_0)}{\theta_c(\theta_0) - \theta_0}, \tag{8}$$

which is the slope of the line connecting θ_0 and $\theta_c(\theta_0)$ on $R(\theta)$.[21]

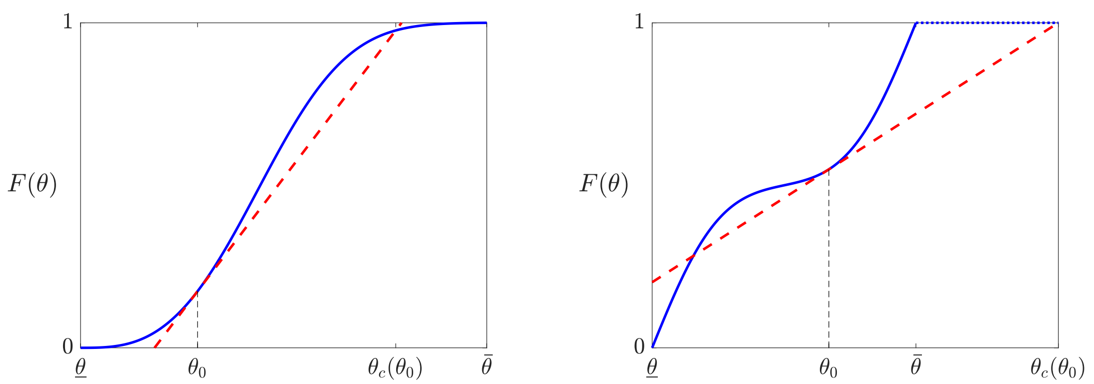
[21] In particular, if $\theta_c(\theta_0) = \theta_0$ (i.e., $\theta_0 = 0$), $A(\theta_0) = \lim_{\theta \to \theta_0^+} \frac{R(\theta) - R(\theta_0)}{\theta - \theta_0} = r(\theta_0+)$.

Example (Quality Maximization)**.** For $v(q,\theta) = q$ and $c(q) = q^2/2$, $A(\theta_0) = \frac{F(2\theta_0)-F(\theta_0)}{\theta_0}$.

Now I state the necessary and sufficient conditions that depend on $r(\theta)$. Although they contain quantifiers that depend on the existence of θ_0, I will provide later conditions that are easy to check.

Condition (S) (Subgradient). There exists some θ_0 such that $\int_{\theta_0}^{\theta} r(x)\,\mathrm{d}x \geq A(\theta_0)\cdot(\theta-\theta_0)$ for all $\theta \in [0, \theta_1(\theta_0)]$.

By the definition of $A(\theta_0)$, condition (S) holds with equality at $\theta = \theta_c(\theta_0)$. Condition (S) says that $A(\theta_0)$ is the *subgradient* of $R|_{[0,\theta_c(\theta_0)]}$ at θ_0. If $R(\theta)$ is differentiable at θ_0, then $r(\theta_0) = A(\theta_0)$. Figure 2 illustrates this condition in the quality maximization case when $R(\theta) = F(\theta)$. The line ℓ connecting θ_0 and $\theta_c(\theta_0)$ (red dashed line) is the supporting hyperplane of $\operatorname{epi} F|_{[0,\theta_c(\theta_0)]}$ at θ_0. If $F(\theta)$ is differentiable at θ_0, then ℓ must be tangent to $F(\theta)$ at θ_0.



(a) $F(\theta)$ that satisfies condition (S) at θ_0 (b) $F(\theta)$ that violates condition (S) at θ_0

Figure 2: Geometric Illustration of Condition (S) when $R(\theta) = F(\theta)$

Condition (C) (Concavity). There exists some θ_0 such that $r(\theta)$ is decreasing in θ on $[\theta_1(\theta_0), \bar{\theta}]$.[22]

To characterize the set of functions that satisfy conditions (S) and (C), I introduce the following definitions that generalize unimodal, increasing, and decreasing functions.

Definition 4. A function $r(\theta)$ is

- *quasi-unimodal* if it satisfies conditions (S) and (C),

[22] In particular, condition (C) implies $r(\theta)$ is decreasing at $\theta_1(\theta_0)$. When $\underline{\theta} > 0$, this rules out the possibility that $\theta_c(\theta_0) \leq \underline{\theta}$ because it would imply $r(\underline{\theta}) \leq r(\underline{\theta}-) = 0$.

- *quasi-increasing* if it satisfies condition (S) at some θ_0 such that $\theta_c(\theta_0) \geq \bar{\theta}$,
- *quasi-decreasing* if it satisfies conditions (S) and (C) at some $\theta_0 \leq \underline{\theta}$.

Loosely speaking, $r(\theta)$ is quasi-unimodal if types are concentrated around the mode of $r(\theta)$ and quasi-increasing (quasi-decreasing) if types are concentrated towards the top (bottom) of $r(\theta)$. While an increasing, decreasing, or unimodal $r(\theta)$ is quasi-increasing, quasi-decreasing, or quasi-unimodal, respectively, the converse does not necessarily hold, as the definitions allow some deviations from monotonicity or unimodality (See Lemma A.3). The magnitude of deviations depends on $[\underline{\theta}, \bar{\theta}]$ and the cost function. For example, if $\underline{\theta} = 0$, then $r(\theta)$ is quasi-decreasing if and only if it is decreasing.

Based on Definition 4, I provide the necessary and sufficient conditions for lower censorship, pass/fail tests, and lower censorship without exclusion.

Proposition 1 (Necessary and Sufficient Conditions)**.** *Under Condition LD, the optimal deterministic rating scheme*

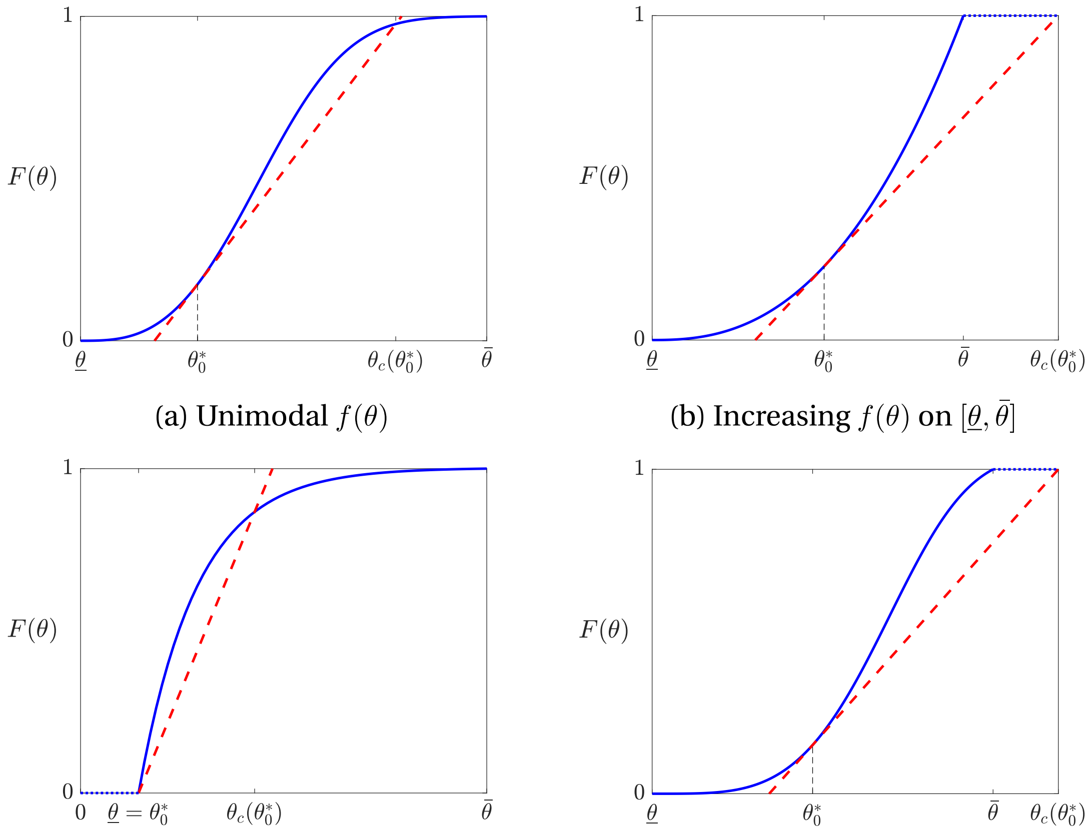
- *is lower censorship (with cutoff type θ_0^*) if and only if $r(\theta)$ is quasi-unimodal (with conditions* (S) *and* (C) *satisfied at θ_0^*),*
- *is pass/fail if and only if $r(\theta)$ is quasi-increasing,*
- *induces no exclusion if and only if $r(\theta)$ is quasi-decreasing,*
- *is fully revealing if and only if $r(\theta)$ is decreasing on* $[0, \bar{\theta}]$.

Example (Quality Maximization)**.** If $v(q, \theta) = q$, then the proposition holds with $r(\theta) = f(\theta)$. Figure 3 illustrates some distributions for which conditions (S) and (C) are satisfied.

Remark 2. In the quality maximization case, Zapechelnyuk (2020, Theorem 2) provides a sufficient condition for pass/fail that is equivalent to $f(\theta)$ being unimodal and $\bar{\theta} \leq \theta_c(\underline{\theta})$, which implies that $f(\theta)$ is quasi-increasing.[23] The assumption that $\bar{\theta} \leq \theta_c(\underline{\theta})$ is restrictive when $\underline{\theta}$ is small because $\theta_c(\underline{\theta}) \searrow 0$ as $\underline{\theta} \searrow 0$.

Remark 3. If $\underline{\theta} = 0$, then lower censorship induces no exclusion if and only if it is fully revealing.

[23] His assumption 3 that $u(q_f(\bar{\theta}), \theta) \geq 0$ for all $\theta \in [\underline{\theta}, \bar{\theta}]$ is equivalent to $q_i(\theta) \geq q_f(\bar{\theta})$ for all $\theta \in [\underline{\theta}, \bar{\theta}]$, which is equivalent to $q_i(\underline{\theta}) \geq q_f(\bar{\theta})$—i.e., $\bar{\theta} \leq \theta_c(\underline{\theta})$.



(a) Unimodal $f(\theta)$ (b) Increasing $f(\theta)$ on $[\underline{\theta}, \bar{\theta}]$

(c) Decreasing $f(\theta)$ on $[\underline{\theta}, \bar{\theta}]$ (d) Unimodal (and quasi-increasing) $f(\theta)$

Figure 3: Quality Maximization

See the proof in Appendix A.2.1.

If conditions (S) and (C) are satisfied at θ_0^*, the optimal deterministic rating has a minimum standard $q_0 = q_i(\theta_0^*)$ above which it fully reveals quality. In particular, if $\theta_c(\theta_0^*) \geq \bar{\theta}$ (or equivalently, $q_i(\theta_0^*) \geq q_f(\bar{\theta})$), the optimal deterministic rating is a pass/fail test.

The following corollary provides sufficient conditions that are easy to check, as they guarantee the existence of θ_0 that satisfies conditions (S) and (C) without solving for it.

Corollary 1.1. *Sufficient conditions for lower censorship, pass/fail tests, and lower censorship without exclusion are that $r(\theta)$ is unimodal, increasing, and decreasing, respectively.*

Figure 4 illustrates optimal quality scheme $q^*(\theta)$ for decreasing, unimodal, and increasing $f(\theta)$ in the quality maximization case. Importantly, the mode θ_m of the density $f(\theta)$ must be in the bunching region $[\theta_0^*, \theta_1(\theta_0^*)]$. In other words, if f is unimodal with mode θ_m, then conditions (S) and (C) will be satisfied at some $\theta_0^* \in [\theta_c^{-1}(\theta_m), \theta_m]$.

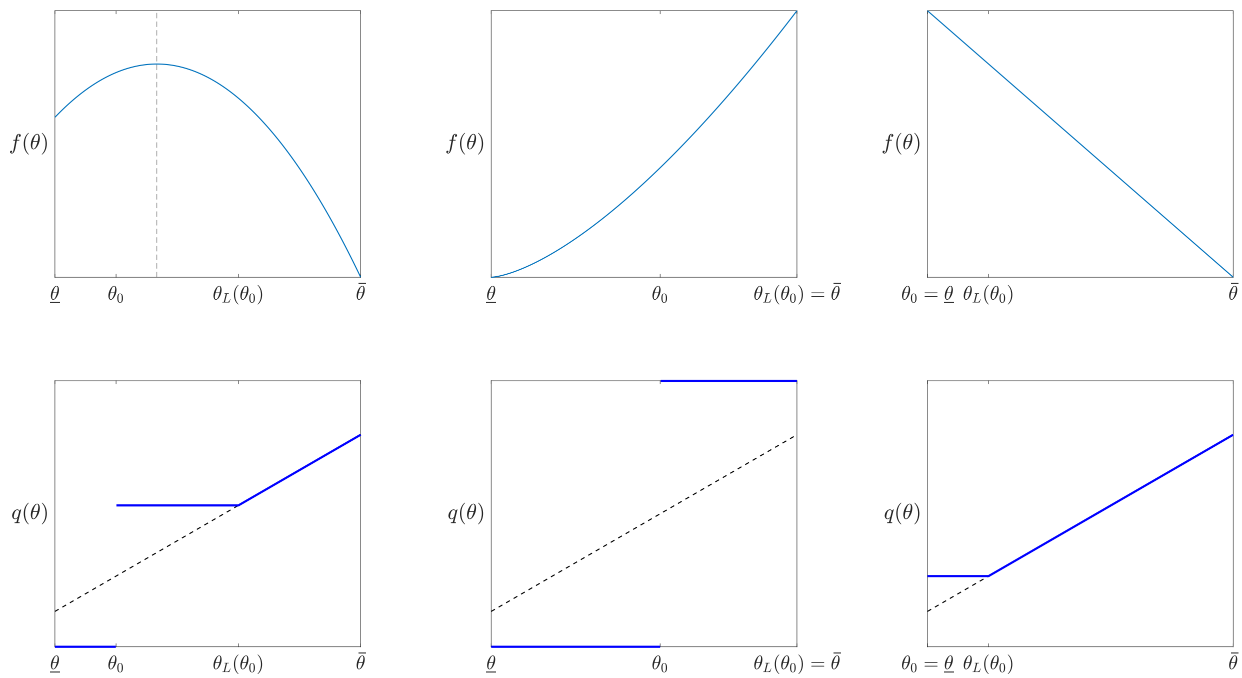


Figure 4: $q^*(\theta)$ for unimodal, increasing, and decreasing $f(\theta)$ in quality maximization

Example 4.1 (Quality Maximization)**.** Assume $v(q, \theta) = q$, $c(q) = q^2/2$, $\Theta = [0, 1]$, and $F(\theta) = \theta^a$. Then, if $a \geq 1$, a pass/fail test is optimal, and the optimal cutoff θ_0^* is given by

$$A(\theta_0^*) \equiv \frac{1 - F(\theta_0^*)}{\theta_0^*} = f(\theta_0^*) \implies \theta_0^* = (1 + a)^{-1/a}.$$

When $a = 1$, $\theta_0^* = 1/2$. As a increases (i.e., F becomes more convex), the optimal cutoff increases to 1. If $a \leq 1$, full revelation ($q^*(\theta) = \theta$) is optimal because $\theta_0^* = \underline{\theta} = 0$.

Intuition for quality maximization. First, consider a perturbation that slightly changes the optimal quality scheme in the fully revealing region at $\hat{\theta} \in (\theta_1(\theta_0^*), \bar{\theta})$. By Lemma F.1, this perturbation leads to

$$\hat{q}(\theta) = \begin{cases} q_f(\hat{\theta} - \varepsilon), & \text{if } \theta \in (\hat{\theta} - \varepsilon, \hat{\theta}), \\ q_f(\hat{\theta} + \varepsilon), & \text{if } \theta \in (\hat{\theta}, \hat{\theta} + \varepsilon). \end{cases}$$

This is induced by a minimum standard at $q_f(\hat{\theta}+\varepsilon)$ that reveals q if and only if $q \geq q_f(\hat{\theta}+\varepsilon)$ in the perturbation region $(q_f(\hat{\theta} - \varepsilon), q_f(\hat{\theta} + \varepsilon))$. This minimum standard creates two pooling regions: $[\hat{\theta}-\varepsilon, \hat{\theta}]$ (lower types) and $[\hat{\theta}, \hat{\theta}+\varepsilon]$ (higher types) and leads to a trade-off: On the one hand, it induces higher types to invest more in quality than they would under full revelation to separate themselves from the lower types who would rather not meet the standard. On the other hand, it discourages the lower types from investing in quality because they would rather bunch at the lower quality level and not reach the minimum standard.

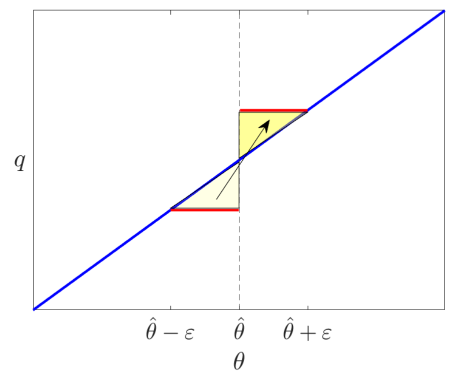


Figure 5: A perturbation to $q_f(\theta)$ in the fully revealing region

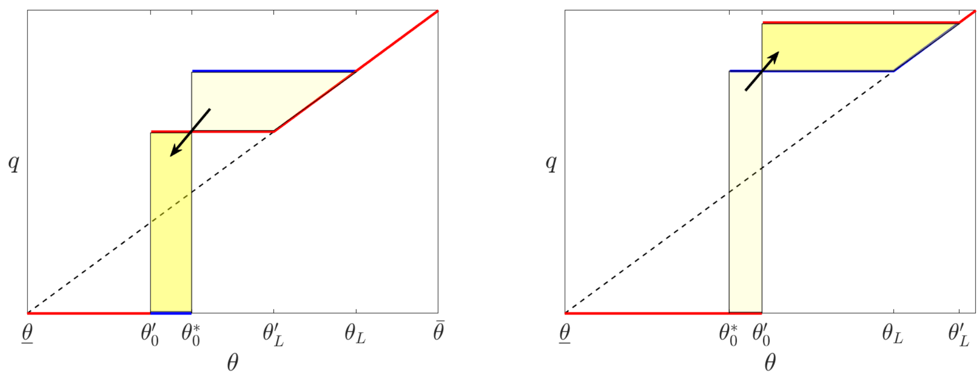
Figure 5 illustrates this trade-off when $c(q) = q^2/2$. In this case, the loss in average quality due to discouraged lower types is represented by the triangle on the left (light yellow), while the gain due to the motivated higher types is represented by the triangle on the right (bright yellow). The two triangles have the same area.[24] Therefore, the shift of the mass of the area from the left to the right decreases average quality if and only if the density $f(\theta)$ is decreasing at $\hat{\theta}$. In other words, the full-revelation quality $q_f(\theta)$ is optimal whenever $f(\theta)$ is decreasing.

On the other hand, if the density $f(\theta)$ is increasing on $[\underline{\theta}, \bar{\theta}]$, the gain from higher types always exceeds the loss from lower types, even as the perturbation becomes large, because an increasing density implies that the mass of the triangle on right is larger than the right. Thus, the optimal rating does not induce any fully revealing region, and agents either choose $q = 0$ or the minimum standard. In other words, if $f(\theta)$ is increasing, the principal will set a high minimum standard such that even the highest type needs to invest more in quality than he would under full revelation to pass the test, in order to provide stronger incentives to high types at the cost of excluding more low types.

Second, consider a perturbation to the optimal scheme in exclusion and bunching regions. The perturbation can involve either a lower or higher minimum standard, which

[24]For general $c(q)$, the loss and the gain regions still have the same area, although not necessarily triangles.

leads to more or less participation. Intuitively, a lower minimum standard increases participation because more lower types can reach the standard without violating their participation constraints. On the other hand, it reduces the incentives for higher types who are bunched at the minimum standard. Analogously, a higher minimum standard reduces participation but increases the incentives for higher types to invest in quality.



(a) More participation

(b) Less participation

Figure 6: Perturbations on pooling regions

Figure 6 illustrates this trade-off in both directions. Similar to a perturbation in the fully revealing region, the loss (in light yellow) and the gain (in bright yellow) have the same area. If $f(\theta)$ is unimodal with a mode $\theta_m \in [\underline{\theta}, \bar{\theta}]$, the optimal cutoff θ_0^* is such that $\theta_m \in [\theta_0^*, \theta_1(\theta_0^*)]$ is in the bunching region. Thus, unimodality of the density implies either more or less participation is undesirable. More generally, if conditions (S) and (C) hold at some θ_0^*, which are implied by unimodality, then θ_0^* is the optimal cutoff type (see Lemma A.2).

In particular, if $f(\theta)$ is decreasing on $[\underline{\theta}, \bar{\theta}]$, no exclusion is optimal because reducing participation for a higher minimum standard (Figure 6b) is undesirable. If $f(\theta)$ is increasing, then $\theta_1(\theta_0^*) = \bar{\theta}$ (because the mode is $\bar{\theta}$), so a pass/fail test is optimal. Similarly, if f is unimodal and $\theta_c(\underline{\theta}) \geq \bar{\theta}$, then $\theta_1(\theta_0^*) = \bar{\theta}$ for every possible $\theta_0^* \in [\underline{\theta}, \bar{\theta}]$, so a pass/fail test is also optimal. This is because the range of types is so small that the start of the fully revealing region is higher than $\bar{\theta}$; consequently, the fully revealing region can never be reached.

Intuition for Linear Delegation. Under Condition LD, the characteristic function $r(\theta)$ incorporates $\beta(\theta)$ and α into the density function $f(\theta)$. First, consider the role of $\beta(\theta)$ by fixing $\alpha = 0$. Then, the objective is $\int_{\underline{\theta}}^{\bar{\theta}} v(q,\theta) f(\theta)\, \mathrm{d}\theta = \int_{\underline{\theta}}^{\bar{\theta}} q(\theta)\beta(\theta) f(\theta)\, \mathrm{d}\theta$, so $\beta(\theta)$ can be easily incorporated into the density $f(\theta)$. In other words, $\tilde{f}(\theta) \equiv \beta(\theta) f(\theta)$ can be treated as the density. Thus, the intuitions for the quality maximization case that relates the density to the optimal deterministic rating scheme carry over.

Now consider the role of α. As α increases from 0 to 1, $r(\theta) = (1-\alpha)\tilde{f}(\theta) - \alpha(F(\theta) - F(\theta_0))$ is more likely to be decreasing. Intuitively, as α increases, the principal's preference becomes more aligned with the agent's, so full revelation is more likely to be optimal.

Comparison with Amador and Bagwell (2022). Now I briefly compare my results with Amador and Bagwell (2022, henceforth AB); a detailed comparison is in Appendix D.

First, AB's condition (i) is stronger than condition (S) because it rules out the possibility that $\theta_c(\theta_0) > \bar{\theta}$ (so that $\theta_1(\theta_0) = \bar{\theta}$). Thus, the condition requires that the fully revealing region $[\theta_1(\theta_0), \bar{\theta}]$ must be nonempty.

Second, AB require condition (C) to hold for *all* $\theta_0 \in [\underline{\theta}, \bar{\theta})$. Consequently, condition (S) can only hold at $\theta_0 \leq \underline{\theta}$, so no exclusion is optimal. Therefore, a pass/fail test can never be optimal (except in the trivial case where no type fails in the equilibrium—i.e., $\bar{\theta} \leq \theta_c(\underline{\theta})$).

4.3.2 Approximation

In reality, a pass/fail test may be preferred because it is simple, although not necessarily optimal. The following result shows that a pass/fail test can guarantee a constant fraction of the maximum expected quality for general thin-tail distributions (relative to the exponential distribution).

Claim 1. *Assume $v(q,\theta) = q$, $c(q) = q^2/2$, and $\underline{\theta} = 0$. If $f(\theta)$ is increasing, a pass/fail test is optimal. If $f(\theta)$ is decreasing, a pass/fail test can still achieve at least $2/e \approx 73.6\%$ of the maximum expected quality if $F(\theta)$ has increasing failure rates (IFR). The bound is tight if F is the exponential distribution.*

If $f(\theta)$ is unimodal, a pass/fail test can achieve at least $(1+e/2)^{-1} \approx 42.4\%$ of the maximum expected quality under IFR.

Example 4.1 (continued). Assume $v(q,\theta) = q$, $c(q) = q^2/2$, $\Theta = [0,1]$, and $F(\theta) = \theta^a$. If $a \leq 1$, $q^*(\theta) = \theta$ is optimal and results in an expected quality of $\mathbf{E}[\theta] = a/(a+1)$. Alternatively, a pass/fail test with the cutoff $\theta_0^* = (1+a)^{-1/a}$ results in an expected quality of $2(1+a)^{-1/a}\,\mathbf{E}[\theta]$. The constant $2(1+a)^{-1/a} \in (2/e, 1]$ is increasing in a on $(0,1]$.

4.3.3 Applications

Now I provide several applications of the necessary and sufficient conditions under linear delegation other than the running example of quality maximization, especially state-dependent preferences.

In many applications, the principal internalizes a fraction $\alpha \in [0,1]$ of the agent's costs, $v(q,\theta) = q - \alpha c(q)/\theta$ (Onuchic and Ray, 2023; Bizzotto and Harstad, 2023).

Example 4.2 (Partial Cost Internalization)**.** Assume $v(q,\theta) = q - \alpha c(q)/\theta$. Then,

$$r(\theta) = (1-\alpha)f(\theta) - \alpha(\tilde{F}(\theta) - \tilde{F}(\theta_0)),$$

where $\tilde{F}(\theta) = \int_{\underline{\theta}}^{\theta} f(x)/x \, \mathrm{d}x$.[25] The function $r(\theta)$ is a weight sum of the density $f(\theta)$ and a decreasing function $-(\tilde{F}(\theta) - \tilde{F}(\theta_0))$.

In the utilitarian benchmark where $\alpha = 1$, because $r(\theta) = -(\tilde{F}(\theta) - \tilde{F}(\theta_0))$ is decreasing on $\mathbb{R}_+$, a fully revealing test is optimal. As α decreases to 0, having a minimum standard becomes optimal because the first term $(1-\alpha)f(\theta)$ matters more and $r(\theta)$ is no longer decreasing on $\mathbb{R}_+$ (unless $\underline{\theta} = 0$). Intuitively, as the preference misalignment increases, it is optimal to have a minimum standard to provide stronger incentives. If $f(\theta)$ is decreasing, the optimal minimum standard will not lead to exclusion because $r(\theta)$ is still decreasing on $[\underline{\theta}, \bar{\theta}]$ and thus quasi-decreasing on $\mathbb{R}_+$. On the other hand, if $f(\theta)$ is unimodal or increasing, tests with minimum standard (lower censorship or pass/fail) that entail exclusion can be optimal as the preferences become more misaligned.

Next, I consider quadratic loss utility functions (with downward bias), a widely studied case in optimal delegation (e.g., Alonso and Matouschek (2008); Kleiner et al. (2021); Kartik et al. (2021)).

Example 4.3 (Quadratic Loss)**.** Assume $v(q,\theta) = -(q-\beta(\theta))^2$ and $u(q,\theta) = -(q-\theta)^2$ with $\beta(\theta) \geq \theta$ and $\Theta = [0,1]$. This is equivalent to $v(q,\theta) = \beta(\theta)q - q^2/2$ and $u(q,\theta) = \theta q - q^2/2$ (i.e., linear delegation with $c(q) = q^2/2$ and $\alpha = 1$). Then,

$$r(\theta) = (\beta(\theta) - \theta)f(\theta) - (F(\theta) - F(\theta_0)).$$

In particular, Proposition 1 implies that full delegation (corresponding to a fully revealing test) is optimal if and only if $r(\theta)$ is decreasing.

4.4 General Preferences

Now I consider the general case where the principal's preference $v(q,\theta)$ does not necessarily satisfy Condition LD (linear delegation). For example, the principal partially

[25] To see this, note that $\int_{\underline{\theta}}^{\bar{\theta}} (q - \alpha c(q)/\theta) \, \mathrm{d}F(\theta) = \int_{\underline{\theta}}^{\bar{\theta}} (\theta q - \alpha c(q)) \, \mathrm{d}\tilde{F}(\theta)$, and that $v(q,\theta) = \theta q - \alpha c(q)$ induces $r(\theta) = (1-\alpha)\theta f(\theta) - \alpha(F(\theta) - F(\theta_0))$.

internalizes the agent's cost—i.e., $v(q,\theta) = \theta q - \tilde{c}(q)$, where $\tilde{c}(q)$ is strictly increasing, convex, and satisfies Assumption 1. When c' is not a linear transformation of $\tilde{c}'$ (i.e., $\tilde{c}''(q)/c''(q)$ is nonconstant), this is *nonlinear* delegation (see Kolotilin and Zapechelnyuk (2025)).

For general preferences, the characteristic functions $r(\theta)$ and $R(\theta)$ can take more general forms. Nevertheless, the conditions in Proposition 1 remain *sufficient* for the optimality of lower censorship and pass/fail tests, with the $r(\theta)$ function in conditions (S) and (C) replaced by a more complicated function.

Definition of $r(\theta)$ for General Preferences. Define the *relative concavity* of the principal and agent's preferences by

$$\kappa = \inf_{q\in Q, \theta\in[\underline{\theta},\bar{\theta}]} \{-v_{qq}(q,\theta)/c''(q)\}. \tag{9}$$

Define

$$r(\theta|q) = v_q(q,\theta)f(\theta) - \kappa(\theta - c'(q))f(\theta) - \kappa(F(\theta) - F(\theta_0)). \tag{10}$$

Slightly abusing the notation, substituting $q(\theta)$ for lower censorship or pass/fail in equation (5) into $r(\theta|q)$, define $r(\theta) = r(\theta|q(\theta))$ and $R(\theta) = \int_{\underline{\theta}}^{\theta} r(\tilde{\theta})\,\mathrm{d}\tilde{\theta}$.[26] As before, by convention, $r(\theta) = \kappa F(\theta_0) \geq 0$ for all $\theta < \underline{\theta}$ and $r(\theta) = -\kappa(1 - F(\theta_0)) \leq 0$ for all $\theta > \bar{\theta}$. Define

$$L(\theta|\theta_0) = \begin{cases} \frac{1}{\theta_0-\theta}\left[\int_{\theta}^{\theta_0} v_q(0,\tilde{\theta})f(\tilde{\theta})\,\mathrm{d}\tilde{\theta} - \kappa\theta(F(\theta_0) - F(\theta))\right], & \text{if } \theta \in [\underline{\theta}, \theta_0), \\ \frac{1}{\theta-\theta_0}\left[\int_{\theta_0}^{\theta} v_q(q_i(\theta_0),\tilde{\theta})f(\tilde{\theta})\,\mathrm{d}\tilde{\theta} - \kappa(\theta - \theta_c(\theta_0))(F(\theta) - F(\theta_0))\right], & \text{if } \theta \in (\theta_0, \theta_c(\theta_0)], \end{cases} \tag{11}$$

which is the slope of the line connecting θ_0 and θ on $R(\theta)$. Define $A(\theta_0) = L(\theta_c(\theta_0)|\theta_0)$.

The following proposition characterizes *sufficient* conditions for lower censorship and pass/fail tests to be optimal for general preferences.

Proposition 2 (Sufficient Conditions)**.** *The optimal deterministic rating scheme*

- *is lower censorship (with cutoff type θ_0^*) if $r(\theta)$ is quasi-unimodal (with conditions* (S) *and* (C) *satisfied at θ_0^*),*
- *is pass/fail if $r(\theta)$ is quasi-increasing,*
- *induces no exclusion if $r(\theta)$ is quasi-decreasing, and*

[26]For all $\theta \geq \bar{\theta}$, we have $r(\theta) = -\kappa(1-F(\theta_0))$ so that $R(\theta) = R(\bar{\theta}) - \kappa(1-F(\theta_0))(\theta - \bar{\theta})$ because $f(\theta) = 0$ and $F(\theta) = 1$. Note that $r(\theta)$ may be discontinuous at θ_0 because $r(\theta_0^-) \geq r(\theta_0^+)$ (see Lemma A.4).

- *is fully revealing if* $r(\theta)$ *is decreasing on* $[0, \bar{\theta}]$.

The stronger sufficient conditions for lower censorship, pass/fail tests, and lower censorship without exclusion are that $r(\theta)$ *is unimodal, increasing, and decreasing, respectively.*

In Appendix F, I also provide sufficient conditions for the optimality of exclusion or "no rent at bottom" (i.e., $\theta_0^* \geq \underline{\theta}$, which implies $U(\underline{\theta}) = 0$) and no exclusion (i.e., $\theta_0^* \leq \underline{\theta}$) for general preferences.

4.5 Beyond Lower Censorship

In Appendix B, I characterize the optimal deterministic rating schemes without restricting attention to lower censorship. For example, if the ability distribution is bimodal, the optimal deterministic rating that maximizes expected quality has at most two thresholds—e.g., high-pass/low-pass/fail.

5 Optimal Stochastic Ratings

5.1 Principal's Problem

In this section, I study the optimal rating design without the restriction to deterministic rating schemes, so that $w(\theta) = q(\theta)$ is no longer necessary. Instead, the following lemma provides a necessary and sufficient condition for the feasibility of an incentive-compatible direct mechanism $(q(\theta), w(\theta))$.

Lemma 5 (Saeedi and Shourideh, 2020, Proposition 1 and Theorem 1)**.** *An incentive-compatible direct mechanism* $(q(\theta), w(\theta))$ *is feasible if and only if* $w(\theta)$ *is a mean-preserving spread of* $q(\theta)$ *in the quantile space, that is,*

$$\int_{\underline{\theta}}^{\theta} w(\theta')\,\mathrm{d}F(\theta') \geq \int_{\underline{\theta}}^{\theta} q(\theta')\,\mathrm{d}F(\theta') \text{ for all } \theta \in [\underline{\theta}, \bar{\theta}]$$

with equality at $\theta = \bar{\theta}$.

The result is reminiscent of the symmetric version of Border's theorem (or Maskin-Riley condition) (Maskin and Riley, 1984; Border, 1991)[27] and allows us to optimize over feasible direct mechanisms $(q(\theta), w(\theta))$ rather than Blackwell experiments themselves.

[27] It can also be proven à la the proof of Border's theorem in Kleiner, Moldovanu, and Strack (2021, Theorem 3).

The principal's problem becomes

$$\max_{q(\theta),w(\theta)} \int_{\underline{\theta}}^{\bar{\theta}} v(q(\theta),\theta)\,\mathrm{d}F(\theta)$$

subject to, for all $\theta \in [\underline{\theta}, \bar{\theta}]$,

$$\int_{\underline{\theta}}^{\theta} w(\theta')\,\mathrm{d}F(\theta') \geq \int_{\underline{\theta}}^{\theta} q(\theta')\,\mathrm{d}F(\theta') \tag{MPS}$$

$$\int_{\underline{\theta}}^{\bar{\theta}} w(\theta)\,\mathrm{d}F(\theta) = \int_{\underline{\theta}}^{\bar{\theta}} q(\theta)\,\mathrm{d}F(\theta) \tag{BP}$$

$$\theta w(\theta) - c(q(\theta)) \geq 0 \tag{IR}$$

$$q(\theta) \text{ increasing} \tag{IC-Mon}$$

$$\theta w(\theta) - c(q(\theta)) = \int_{\underline{\theta}}^{\theta} w(x)\,\mathrm{d}x + \underline{U}, \tag{IC-Env}$$

where $\underline{U} = \theta w(\underline{\theta}) - c(q(\underline{\theta}))$.

5.2 When are deterministic ratings optimal?

I focus on the quality maximization case $v(q,\theta) = q$ and provide sufficient conditions for pass/fail and fully revealing tests to be optimal, which depend only on the density $f(\theta)$. The results for general preferences are in Appendix B.2.

Proposition 3. *Assume $v(q,\theta) = q$. The optimal rating scheme is*

- *a pass/fail test if $f(\theta)$ is increasing.*[28]
- *a fully revealing test if and only if both $f(\theta)$ and $\theta f'(\theta)/f(\theta)$ are decreasing and $\underline{\theta} = 0$.*

Remark 4. $\varepsilon_f(\theta) = -\theta f'(\theta)/f(\theta)$ is the *elasticity* of the density $f(\theta)$. When f is decreasing, its elasticity $\varepsilon_f(\theta)$ is increasing if F satisfies the IFR property. The elasticity $\varepsilon_f(\theta)$ is increasing even for distributions that violate the IFR property or Myerson's regularity (e.g., log-normal and Pareto distributions).

Example 5.1 (Pareto Distribution)**.** The Pareto distribution Par(a,b) has a strictly decreasing density $f(\theta) = ab^a\theta^{-(a+1)}$ and a *constant* elasticity $\varepsilon_f(\theta) = a+1$. The condition can be violated by distributions that have heavier tails than the Pareto distribution (e.g., $f(\theta) = \exp(1/\theta)$).

[28]Another sufficient condition is both $\theta_0^* = \underline{\theta}$ (i.e., no exclusion) and $\bar{\theta} \leq \theta_c(\underline{\theta})$ (i.e., variation in types is not large enough to sustain a fully revealing region).

5.3 When does principal benefit from stochastic ratings?

Since stochastic ratings expand the set of incentive-compatible quality $q(\theta)$, a natural question is when stochastic ratings are optimal. The following proposition addresses this question with a sufficient condition.

Proposition 4. *The principal strictly benefits from stochastic rating schemes if the quality scheme induced by the optimal deterministic rating scheme has a fully revealing region in which* $\varepsilon_f(\theta) = -\theta f'(\theta)/f(\theta)$ *is not increasing.*

Intuitively, stochastic rating schemes can allow $\hat{w}'(q) > 1$ for some qualities (and therefore $w'(\theta) > q'(\theta)$ for some types) to provide stronger incentives than fully revealing any marginal investment in quality to the market. This can be achieved, for example, by increasing the probability of the agent's quality being pooled with higher qualities (or separated from lower qualities). Consequently, this partial pooling leads to higher $q(\theta)$ for some (lower) types at the expense of lower $q(\theta)$ for other (higher) types, which can be more desirable for the principal under heavy-tail distributions.

6 Conclusion

Ratings are often used to motivate agent performance or firm investment in product quality, particularly when monetary transfers are limited. When the market rewards agents based on the perception of their endogenous quality or exogenous abilities, ratings can provide reputational incentives in place of monetary incentives. In this paper, I study the optimal rating scheme to incentivize agents' investment in quality when they have private information about their costs of investment.

By defining an interim wage function and characterizing necessary and sufficient conditions for an incentive-compatible direct mechanism to be feasible (i.e., can be induced by a rating scheme), I use an interim approach to the rating design problem. The interim approach is particularly useful in solving for the optimal general (possibly stochastic) rating scheme, as it reduces the rating design problem to the optimization over interim wage functions rather than ratings themselves.

I provide necessary and sufficient conditions under which pass/fail tests and lower censorship are optimal among deterministic ratings. In particular, when the principal's objective is expected quality, lower censorship is optimal if and only if types are concentrated around the mode of the distribution (i.e., density is quasi-unimodal), and pass/fail tests are optimal if types are concentrated towards the top (i.e., density is quasi-increasing). Beyond lower censorship, I also solve for the optimal deterministic ratings for general preferences and distributions. In the quality maximization case, the optimal deterministic rating can take the form of high-pass/low-pass/fail if the ability distribution is bimodal.

The deterministic rating design problem is equivalent to a delegation problem with outside option (Amador and Bagwell, 2022). My results improve upon the existing results by providing weaker sufficient conditions for lower censorship (corresponding to threshold delegation) that are also necessary in the linear delegation case. I also provide necessary and sufficient conditions for pass/fail tests (corresponding to take-it-or-leave-it offers or bang-bang allocations in delegation) to be optimal. The results allow for general state-dependent preferences of the principal and nonlinear delegation. Additionally, through the equivalence established by Kolotilin and Zapechelnyuk (2025), the results also have implications for the Bayesian persuasion literature, especially in cases where the sender's payoffs are nonlinear in the state.

When stochastic rating schemes are allowed, I also provide sufficient conditions under which pass/fail tests remain optimal. In the quality maximization case, a pass/fail test is optimal if the ability density is increasing. Moreover, I identify conditions under which stochastic ratings strictly improve on deterministic ratings. For example, a noisy test that partially pools low quality with high quality enables the principal to increase the incentives for low types at the cost of incentives for high types, which can increase the overall expected quality if the ability density has a heavier tail than the Pareto distribution—in other words, they are a few very high-ability agents.

Nevertheless, I have not characterized the optimal ratings in general when stochastic ratings are feasible. Further, in the current model, the market either values the agent's endogenous quality or exogenous abilities, but a combination of both cases is not considered. One would expect a combination of them makes the full revelation of quality more likely to be optimal than the former and less likely than the latter. Moreover, while I focus on the case where agents can choose quality deterministically, the more general case where investing effort increases quality stochastically is worth exploring. In addition, competition among test designers is a direction for future research.

Appendix A Proofs

A.1 Proofs of Sections 3

Proof of Lemma 2. ($\Longrightarrow$) is by the revelation principle and the definition of feasibility. ($\Longleftarrow$) is similar to the taxation principle. Construct a $\pi(q)$ that penalizes off-path deviations to q that no types choose in the direct mechanism, so that they will never be chosen in the rating scheme $\pi(q)$ either. $\square$

Lemma A.1 (Incentive Compatibility)**.** *A direct mechanism* $(q(\theta), w(\theta))$ *is incentive compatible if and only if*

- $w(\theta)$ *is increasing, and*
- $U(\theta) \equiv \theta w(\theta) - c(q(\theta)) = \int_{\underline{\theta}}^{\theta} w(x)\,\mathrm{d}x + \underline{U}$,[29]

where $\underline{U} = \underline{\theta} w(\underline{\theta}) - c(q(\underline{\theta}))$*. The first condition can be replaced by the monotonicity of* $q(\theta)$ *(i.e.,* $q(\theta)$ *is increasing).*

Proof. Proof is standard by noting that $U(\theta) = \max_{\hat{\theta}}\{\theta w(\hat{\theta}) - c(q(\hat{\theta}))\}$. $\square$

A.2 Proofs of Sections 4

Proof of Lemma 3. Under a deterministic rating scheme π, if the rating maps a (potentially singleton) nonempty set of quality to the same score s, only $\hat{q}(s) \equiv \min\{q : \pi(q) = s\}$ will be chosen by an agent. Thus, for any $q \in \{\hat{q}(s) : s \in \pi(Q)\}$ (where $\pi(Q) \equiv \{\pi(q) : q \in Q\}$) chosen by an agent, the interim wage is $\hat{w}(q) = \mathbf{E}[\tilde{q} \mid s = \pi(q)] = q$. Therefore, for any $\theta \in [\underline{\theta}, \bar{\theta}]$, the interim wage is $w(\theta) \equiv \hat{w}(q(\theta)) = q(\theta)$. $\square$

Proof of Lemma 4. By (IR) and (IC), there exists a cutoff type $\theta_0 \in [\underline{\theta}, \bar{\theta}]$ such that $U(\theta) \geq 0$ if and only if $\theta \geq \theta_0$. If $\theta < \theta_0$, then $U(\theta) < 0$, so the agent chooses $q(\theta) = 0$. If $\theta > \theta_0$, then $U(\theta) < 0$ and thus $q(\theta) > 0$. If $\theta_0 \in (\underline{\theta}, \bar{\theta})$ is in the interior, then $U(\theta_0) = 0$, so the agent is indifferent between $q_i(\theta_0)$ and $q = 0$. $\square$

Proof of Claim 1. (i) When $c(q) = q^2/2$ and $\underline{\theta} = 0$, if $f(\theta)$ is decreasing, the optimal rating scheme is fully revealing that induces $q_f(\theta) = \theta$, while a pass/fail test with cutoff type θ_0 induces

$$q(\theta) = \begin{cases} 2\theta_0, & \text{if } \theta \geq \theta_0 \\ 0, & \text{otherwise} \end{cases}$$

[29]For general $c(q, \theta)$, the condition becomes $U(\theta) \equiv w(\theta) - c(q(\theta), \theta) = -\int_{\underline{\theta}}^{\theta} c_\theta(q(x), x) dx + \underline{U}$, where $\underline{U} = w(\underline{\theta}) - c(q(\underline{\theta}), \underline{\theta})$.

Thus, it suffices to show that $\max_{\theta_0} 2\theta_0(1 - F(\theta_0)) \geq (2/e)\,\mathbf{E}[\theta]$ if F satisfies IFR.

By Theorem 4.4 in Barlow and Proschan (1996, Chapter 2), If F satisfies IFR with mean $\mathbf{E}[\theta]$, then

$$1 - F(\theta) \geq \begin{cases} \exp(-\theta/\,\mathbf{E}[\theta]), & \text{if } \theta < \mathbf{E}[\theta], \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, $\max \theta_0(1 - F(\theta_0)) \geq \max \theta_0 \exp(-\theta_0/\,\mathbf{E}[\theta]) = \mathbf{E}[\theta]/e$. The exponential distribution attains the lower bound.

(ii) If $f(\theta)$ is unimodal, because lower censorship is optimal, the maximal expected quality is

$$\mathbf{E}[q^*(\theta)] = \max_{\theta_0}\left(2\theta_0(F(2\theta_0) - F(\theta_0)) + \int_{2\theta_0}^{\bar{\theta}} \theta\,\mathrm{d}F(\theta)\right) < \max_{\theta_0} 2\theta_0(1 - F(\theta_0)) + \mathbf{E}[\theta].$$

Because $\max_{\theta_0} 2\theta_0(1-F(\theta_0)) \geq (2/e)\,\mathbf{E}[\theta]$, we have $\max_{\theta_0} 2\theta_0(1-F(\theta_0)) > (1+e/2)^{-1}\,\mathbf{E}[q^*(\theta)]$. The bound is not tight because of the strict inequality. □

A.2.1 Proof of Proposition 1

Preliminaries. First, I write $r(\theta)$ in the general form:

$$r(\theta) = \begin{cases} v_q(0,\theta)f(\theta) - \kappa\theta f(\theta) - \kappa(F(\theta) - F(\theta_0)) \\ v_q(q_i(\theta_0),\theta)f(\theta) - \kappa(\theta - \theta_c(\theta_0))f(\theta) - \kappa(F(\theta) - F(\theta_0)) \\ v_q(q_f(\theta),\theta)f(\theta) - \kappa(F(\theta) - F(\theta_0)). \end{cases}$$

Denote by $V(\theta_0)$ the principal's expected payoff given a cutoff type $\theta_0 \in [\underline{\theta}, \bar{\theta}]$, which is given by

$$V(\theta_0) = \int_{\theta_0}^{\theta_1(\theta_0)} v(q_i(\theta_0),\theta)\,\mathrm{d}F(\theta) + \int_{\theta_1(\theta_0)}^{\bar{\theta}} v(q_f(\theta),\theta)\,\mathrm{d}F(\theta),$$

Lemma A.2. *If condition (S) holds at some $\theta_0 \geq 0$, then θ_0 satisfies the first-order condition*[30]

$$V'(\theta_0) = A(\theta_0)q_i(\theta_0) - v(q_i(\theta_0),\theta_0)f(\theta_0) = 0 \qquad \text{(OPT)}$$

In words, increasing the cutoff θ_0 leads to a marginal increase in $A(\theta_0) \cdot q_i(\theta_0)$ in the bunching region (due to a higher minimum standard) and a marginal decrease in the principal's payoff of $v(q_i(\theta_0),\theta_0)f(\theta_0)$ in the exclusion region (due to more exclusion).

[30]When V is non-differentiable at $\theta_0 = \underline{\theta}$ (due to the discontinuity of f), equation (OPT) should take the more general form $0 \in \partial(-V)(\theta_0)$, where $\partial(-V)(\theta_0)$ denotes the subgradient of $-V$ at θ_0 locally.

Proof of Lemma A.2. First, I rewrite $A(\theta_0)$ in an equivalent form. By definition,

$$A(\theta_0) = \frac{1}{\theta_c(\theta_0) - \theta_0} \int_{\theta_0}^{\theta_c(\theta_0)} r(\tilde{\theta}) \, \mathrm{d}\tilde{\theta} = \frac{1}{\theta_c(\theta_0) - \theta_0} \int_{\theta_0}^{\theta_1(\theta_0)} v_q(q_i(\theta_0), \theta) \, \mathrm{d}F(\theta).$$

Hence, the derivative is

$$\begin{aligned} V'(\theta_0) &= \frac{q_i(\theta_0)}{\theta_c(\theta_0) - \theta_0} \int_{\theta_0}^{\theta_1(\theta_0)} v_q(q_i(\theta_0), \theta) \, \mathrm{d}F(\theta) - v(q_i(\theta_0), \theta_0) f(\theta_0) \\ &= A(\theta_0) q_i(\theta_0) - v(q_i(\theta_0), \theta_0) f(\theta_0). \end{aligned}$$

Now it suffices to show that condition (S) holds at some θ_0 implies $V'(\theta_0) = 0$ (OPT). Recall that $r(\theta) = (\beta(\theta) - \alpha\theta) f(\theta) - \alpha(F(\theta) - F(\theta_0))$. Thus,

$$v(q_i(\theta), \theta) f(\theta) = [\beta(\theta) q_i(\theta) - \alpha c(q_i(\theta))] f(\theta) = [r(\theta) + \alpha(F(\theta) - F(\theta_0))] \, q_i(\theta)$$

because $c(q_i(\theta)) = \theta q_i(\theta)$. Hence, $v(q_i(\theta_0), \theta_0) f(\theta_0) = r(\theta_0) q_i(\theta_0)$, and

$$V'(\theta_0) = A(\theta_0) q_i(\theta_0) - v(q_i(\theta_0), \theta_0) f(\theta_0) = (A(\theta_0) - r(\theta_0)) q_i(\theta_0).$$

If condition (S) holds at some $\theta_0 \neq \underline{\theta}$, then $A(\theta_0) = r(\theta_0)$, so $V'(\theta_0) = 0$. If condition (S) holds at $\theta_0 = \underline{\theta}$, then condition (S) implies $A(\underline{\theta}) \in \partial R(\underline{\theta}) = [r(\underline{\theta}-), r(\underline{\theta}+)]$. Thus, $0 \in \partial(-V)(\underline{\theta}) = [V'(\underline{\theta}+), V'(\underline{\theta}-)]$. □

Proof of Proposition 1. **(Sufficiency).** I use the optimal control method to show that the quality scheme $q^*(\theta)$ in equation (5) induced by pass/fail tests or lower censorship is optimal.

Setup of the Hamiltonian

Define $U(\theta) = \int_{\underline{\theta}}^{\theta} q(x) dx + \underline{U}$. Rewrite the constraints as

$$\theta q(\theta) - c(q(\theta)) = U(\theta) \tag{A.1}$$

$$\dot{U} = q(\theta) \tag{A.2}$$

$$\dot{q} = \nu(\theta) \geq 0 \quad (q \text{ increasing}) \tag{A.3}$$

$$U(\underline{\theta}), q(\underline{\theta}) \geq 0, \quad U(\bar{\theta}), q(\bar{\theta}) \text{ free.} \tag{A.4}$$

Set up the Hamiltonian

$$H = v(q(\theta), \theta)f(\theta) + \gamma(\theta)[\theta q(\theta) - c(q(\theta)) - U(\theta)] + \Gamma(\theta)q(\theta) + \mu(\theta)\nu(\theta) \tag{A.5}$$

where U, q are state variables and ν is the control variable; Γ is Hamiltonian multiplier on $\dot{U}$ and μ is Hamiltonian multiplier on $\dot{q}$; γ is the Lagrangian multiplier on $U = \theta q - c(q)$.[31]

By the Pontryagin's maximum principle (Hellwig, 2010, Theorem 4.1), the necessary conditions are

$$-\frac{\partial H}{\partial q} = -(v_q f + \gamma(\theta - c'(q)) + \Gamma) = \dot{\mu} \tag{A.6}$$

$$-\frac{\partial H}{\partial U} = \gamma = \dot{\Gamma} \tag{A.7}$$

$$\frac{\partial H}{\partial \nu} = \mu \leq 0, \quad \mu(\theta) = 0 \text{ if } q \text{ is strictly increasing at } \theta,^{32} \tag{A.8}$$

$$\Gamma(\underline{\theta}) \leq 0, \quad \Gamma(\underline{\theta})U(\underline{\theta}) = 0 \tag{A.9}$$

$$\mu(\underline{\theta}) \leq 0, \quad \mu(\underline{\theta})q(\underline{\theta}) = 0 \tag{A.10}$$

$$\Gamma(\bar{\theta}) = 0, \quad \mu(\bar{\theta}) = 0. \tag{A.11}$$

In the fully revealing region where $q(\theta) = q_f(\theta)$, because $c'(q_f(\theta)) = \theta$ (and thus $\dot{q}_f(\theta) > 0$), we have $\Gamma(\theta) = -v_q(q_f(\theta), \theta)f(\theta)$.

At the cutoff θ_0, the switching condition (Bryson and Ho, 1975, Chapter 3.7) (see also Clarke, 2013, Chapter 22.5 for the hybrid maximum principle)

$$\Gamma(\theta_0+) = \Gamma(\theta_0-) \tag{A.12}$$

$$H(\theta_0+) = v(q_i(\theta_0), \theta_0)f(\theta_0) + \Gamma(\theta_0+)q_i(\theta_0) = H(\theta_0-) = v(0, \theta_0)f(\theta_0) = 0 \tag{A.13}$$

Proposed Multipliers

Given θ_0^*, I propose the following multipliers for the Hamiltonian

$$\Gamma(\theta) = \begin{cases} -A(\theta_0^*) - \kappa(F(\theta) - F(\theta_0^*)), & \text{if } \theta \in [0, \theta_1(\theta_0^*)] \\ -v_q(q_f(\theta), \theta)f(\theta), & \text{if } \theta \in (\theta_1(\theta_0^*), \bar{\theta}) \\ 0, & \text{if } \theta = \bar{\theta} \end{cases} \tag{A.14}$$

[31]Note $U = \theta q - c(q)$ is a pure state constraint (i.e., containing no control variable). Therefore, the multipliers Γ and μ can be discontinuous at junction points between intervals on which the pure state constraint is binding and intervals on which it is not (Seierstad and Sydsaeter, 1977).

[32]A function q is *strictly* increasing at θ if $q(\theta + \varepsilon) - q(\theta + \varepsilon) > 0$ for all $\varepsilon > 0$.

$$\mu(\theta)=\begin{cases}R(\theta_0^*)-R(\theta)-(\theta_0^*-\theta)A(\theta_0^*)\leq 0, & \text{if } \theta\in[0,\theta_1(\theta_0^*)]\\ 0, & \text{if } \theta\in(\theta_1(\theta_0^*),\bar{\theta}]\end{cases} \tag{A.15}$$

By condition (S), in the pooling regions $(\theta_0^*,\theta_1(\theta_0^*))$ and $(\underline{\theta},\theta_0^*]$ (where $q^*(\theta)$ is constant), we have $\mu(\theta)\leq 0$. At the cutoff θ_0^* where $q^*(\theta)$ is strictly increasing, we have $\mu(\theta_1(\theta_0^*))=0$ by condition (S). In the fully revealing region $(\theta_1(\theta_0^*),\bar{\theta}]$ where $q^*(\theta)=q_f(\theta)$ is strictly increasing, we have $\mu(\theta)=0$.

Moreover, by Lemma A.2, θ_0^* satisfies the switching condition

$$H(\theta_0^*+)=v(q_i(\theta_0^*),\theta_0^*)f(\theta_0^*)-A(\theta_0^*)q_i(\theta_0^*)=H(\theta_0^*-)=0.$$

Concavity/Sufficiency

By Kamien and Schwartz (1971), the necessary conditions are sufficient if the maximized Hamiltonian $\bar{H}(q,U,\gamma,\mu,\Gamma)\equiv\max_\nu H(q,U,\nu,\gamma,\mu,\Gamma)$ is concave in state variables (q,U) for given (γ,μ,Γ), which holds if $v_{qq}(q,\theta)f(\theta)-\gamma c''(q)\leq 0$. Recall that $\kappa=\inf_{q,\theta}\{-v_{qq}(q,\theta)/c''(q)\}$, so concavity is satisfied if $\Gamma+\kappa F$ is increasing.

Condition (C) implies $\Gamma+\kappa F$ is increasing on $(\theta_1(\theta_0^*),\bar{\theta}]$ (i.e., fully revealing region). The jumps of $\Gamma(\theta)$ need to be nonnegative at $\theta_1(\theta_0^*)$ and $\bar{\theta}$. At $\theta_1(\theta_0^*)$, there are three cases.

(i) $\theta_1(\theta_0^*)\in(\underline{\theta},\bar{\theta})$ or $\theta_0^*=\underline{\theta}=0$ (so that $\theta_1(\theta_0^*)=\theta_c(\theta_0^*)$). By condition (S), $A(\theta_0^*)=L(\theta_1(\theta_0^*)|\theta_0^*)\geq r(\theta_1(\theta_0^*))=v_q(q_f(\theta),\theta)f(\theta)-\kappa(F(\theta)-F(\theta_0^*))$.

(ii) $\theta_1(\theta_0^*)=\bar{\theta}$. This is implied by $A(\theta_0)\geq 0=\Gamma(\bar{\theta})$.

(iii) $\theta_1(\theta_0^*)=\underline{\theta}>0$. Then $\theta_0^*\leq\underline{\theta}$ and $F(\theta_0^*)=0$. By condition (C), $r(\underline{\theta})=v_q(q_f(\underline{\theta}),\underline{\theta})f(\underline{\theta})\leq r(\underline{\theta}-)=0$, which implies $v_q(q_f(\underline{\theta}),\underline{\theta})f(\underline{\theta})\leq 0\leq A(\theta_0^*)$.

Assumption 1 (i.e., $v_q(q_f(\theta),\theta)\geq 0$) implies the jump of $\Gamma(\theta)$ at $\bar{\theta}$ is nonnegative.

(Necessity). First, I show that condition (S) is necessary. For the quality scheme $q^*(\theta)$ induced by lower censorship or pass/fail tests, the optimal cutoff θ_0^* must satisfy $0\in\partial(-V)(\theta_0^*)$, that is, $V(\theta)-V(\theta_0)\leq 0$. By equation (A.1) in the proof of Lemma A.2,

$$V'(\theta_0)=(A(\theta_0)-r(\theta_0))q_i(\theta_0).$$

Thus,

$$V(\theta)-V(\theta_0^*)=\int_{\theta_0^*}^{\theta}[(A(x)-A(\theta_0^*))+(A(\theta_0^*)-r(x))]q_i(x)\,\mathrm{d}x\leq 0,$$

which implies that $\int_{\theta_0^*}^{\theta}(A(\theta_0^*) - r(x))q_i(x)\,\mathrm{d}x \leq 0$ because the first term $(A(x) - A(\theta_0^*))$ in the integral is positive. Finally, because $q_i(x) > 0$ is increasing, this implies

$$\int_{\theta_0^*}^{\theta}(A(\theta_0^*) - r(x))\,\mathrm{d}x \leq 0,$$

which implies condition (S).

Now I show that condition (C) is necessary for the optimality of the fully revealing region where $q^*(\theta) = q_f(\theta)$. The conditions in Pontryagin's maximum principle are also necessary for optimality. Because q is also a control variable (for $\dot{U} = q$), the second-order necessary condition (i.e., Legendre condition) requires the Hamiltonian be concave in q. On the fully revealing region where $q^*(\theta) = q_f(\theta)$, this implies $v_{qq}(q_f(\theta), \theta)f(\theta) - \gamma c''(q_f(\theta)) \leq 0$. Under Condition LD (linear delegation), because $-v_{qq}(q, \theta)/c''(q) = \alpha$, this necessary condition implies that $r(\theta) = v_q(q_f(\theta), \theta)f(\theta) - \alpha(F(\theta) - F(\theta_0))$ is decreasing on the fully revealing region (condition (C)). □

Proof of Corollary 1.1. I first present the following lemma, which is intuitive by looking at Figure 3. A formal proof is tedious and deferred to Appendix F.

Lemma A.3. *If f is unimodal on $[\underline{\theta}, \bar{\theta}]$, then it is quasi-unimodal. If f is increasing on $[\underline{\theta}, \bar{\theta}]$, then it is quasi-increasing. If f is decreasing on $[\underline{\theta}, \bar{\theta}]$, then it is quasi-decreasing; the converse is true if $\underline{\theta} = 0$. If $\bar{\theta} \leq \theta_c(\underline{\theta})$, then every unimodal $f(\theta)$ is quasi-increasing.*

Corollary 1.1 follows immediately from Proposition 1 and Lemma A.3. □

A.2.2 Proof of Proposition 2

Proof of Proposition 2. First, I show that the point θ_0 at which conditions (S) and (C) hold coincide with the optimal cutoff that satisfies equation (OPT).

Lemma A.4. $r(\theta_0^-) \geq r(\theta_0^+)$ *for all* $\theta_0 \in (\underline{\theta}, \bar{\theta})$*. The equality holds if and only if* $v_{qq}(q, \theta_0) + \kappa c''(q) = 0$ *for almost every* $q \in (0, q_i(\theta_0))$.

If $\theta_0 > \underline{\theta}$*, then condition* (S) *implies* $r(\theta_0+) = r(\theta_0-) = A(\theta_0)$ *and* $v_{qq}(q, \theta_0) + \kappa c''(q) = 0$ *for almost every* $q \in (0, q_i(\theta_0))$.

Proof of Lemma A.4. $r(\theta_0+) = v_q(q_i(\theta_0), \theta_0)f(\theta_0) - \kappa f(\theta_0)(\theta_0 - \theta_c(\theta_0))$ for all $\theta_0 < \bar{\theta}$. $r(\theta_0-) = v_q(0, \theta_0)f(\theta_0) - \kappa f(\theta_0)\theta_0$ for all $\theta_0 > \underline{\theta}$. $r(\theta_0+) \leq r(\theta_0-)$ follows from $v_{qq}(q, \theta_0) + \kappa c''(q) \leq 0$ on $q \in (0, q_i(\theta_0))$ (because $\kappa = \inf\{-v_{qq}/c''(q)\}$); the equality holds if and only if $v_{qq}(q, \theta_0) + \kappa c''(q) = 0$ for almost every $q \in (0, q_i(\theta_0))$.

If $\theta_0 > \underline{\theta}$, then conditions (S) implies $r(\theta_0+) = L(\theta_0+|\theta_0) \geq L(\theta_0-|\theta_0) = r(\theta_0-)$. By Lemma A.4, we must have $r(\theta_0+) = r(\theta_0-) = A(\theta_0)$. □

Lemma A.5. *Conditions* (S) *and* (C) *hold at* θ_0 *only if* θ_0 *satisfies equation* (OPT).

Proof of Lemma A.5. If $\theta_0 > \underline{\theta}$, condition (S) imply $L(\theta_0+|\theta_0) \geq L(\theta_0-|\theta_0)$, so by Lemma A.4, we have $L(\theta_0+|\theta_0) = L(\theta_0-|\theta_0) = A$ and $v_{qq}(q,\theta_0) + \kappa c''(q) = 0$ for almost every $q \in (0, q_i(\theta_0))$. Thus, $A = L(\theta_0+|\theta_0) = (\frac{v(q_i(\theta_0),\theta_0)+\kappa c(q_i(\theta_0))}{q_i(\theta_0)} - \kappa\theta_0)f(\theta_0) = \frac{v(q_i(\theta_0),\theta_0)}{q_i(\theta_0)}f(\theta_0)$, so $V'(\theta_0) = A \cdot q_i(\theta_0) - v(q_i(\theta_0),\theta_0)f(\theta_0) = 0$ (OPT). □

By Lemma A.5, θ_0^* is the optimal cutoff that satisfies the switching condition

$$H(\theta_0+) = v(q_i(\theta_0),\theta_0)f(\theta_0) + \Gamma(\theta_0+)q_i(\theta_0) = H(\theta_0-) = 0,$$

where $\Gamma(\theta_0+) = -A(\theta_0)$. The rest of the proof is the same as the sufficiency part of the proof of Proposition 1 in Appendix A.2.1. □

A.3 Proofs of Section 5

Setup of the Hamiltonian

Define $D(\theta) = \int_{\underline{\theta}}^{\theta}(w(\theta') - q(\theta'))\,\mathrm{d}F(\theta') \geq 0$ and $U(\theta) = \int_{\underline{\theta}}^{\theta} w(x)dx + \underline{U}$. Rewrite the constraints as

$$D(\theta) \geq 0 \text{ (MPS)} \tag{A.16}$$

$$\dot{D} = (w(\theta) - q(\theta))f(\theta) \tag{A.17}$$

$$\theta w(\theta) - c(q(\theta)) = U(\theta) \tag{A.18}$$

$$\dot{U} = w(\theta) \tag{A.19}$$

$$\dot{q} = \nu \geq 0 \quad (q \text{ increasing}) \tag{A.20}$$

$$U(\underline{\theta}), q(\underline{\theta}) \geq 0,\ D(\underline{\theta}) = 0,\ U(\bar{\theta}), q(\bar{\theta}) \text{ free},\ D(\bar{\theta}) = 0 \text{ (BP)} \tag{A.21}$$

Set up the Hamiltonian

$$\begin{aligned} H = v(q(\theta),\theta)f(\theta) + \gamma(\theta)[\theta w(\theta) - c(q(\theta)) - U(\theta)] + \lambda(\theta)D(\theta) \\ +\Lambda(\theta)[w(\theta) - q(\theta)]f(\theta) + \Gamma(\theta)w(\theta) + \mu(\theta)\nu(\theta) \end{aligned} \tag{A.22}$$

where U, q, D are the state variables and w, ν are the control variables; $\lambda(\theta)$ is the Lagrangian multiplier on $D(\theta) \geq 0$ (MPS), $\gamma(\theta)$ is the Lagrangian multiplier on $U(\theta) =$

$\theta w(\theta) - c(q(\theta))$, Λ is the Hamiltonian multiplier on $\dot{D} = [w(\theta) - q(\theta)]f(\theta)$, and Γ is the Hamiltonian multiplier on $\dot{U} = w(\theta)$.

By the Pontryagin's maximum principle, the necessary conditions are

$$-\frac{\partial H}{\partial q} = -(v_q f - \gamma c'(q) - \Lambda f) = \dot{\mu} \tag{A.23}$$

$$-\frac{\partial H}{\partial D} = -\lambda = \dot{\Lambda} \tag{A.24}$$

$$-\frac{\partial H}{\partial U} = \gamma = \dot{\Gamma} \tag{A.25}$$

$$\frac{\partial H}{\partial w} = \theta\gamma + \Lambda f + \Gamma = 0 \tag{A.26}$$

$$\frac{\partial H}{\partial \nu} = \mu \leq 0, \quad \mu(\theta) = 0 \text{ if } q \text{ is strictly increasing at } \theta \tag{A.27}$$

$$\lambda(\theta) \geq 0, \quad \lambda(\theta)D(\theta) = 0 \tag{A.28}$$

$$\Gamma(\underline{\theta}) \leq 0, \quad \Gamma(\underline{\theta})U(\underline{\theta}) = 0 \tag{A.29}$$

$$\mu(\underline{\theta}) \leq 0, \quad \mu(\underline{\theta})q(\underline{\theta}) = 0 \tag{A.30}$$

$$\Gamma(\bar{\theta}) = 0, \quad \mu(\bar{\theta}) = 0 \tag{A.31}$$

$$\Lambda(\bar{\theta}) \text{ no condition.} \tag{A.32}$$

The conditions imply

$$[\theta\Gamma(\theta)]' = \theta\gamma + \Gamma = -\Lambda(\theta)f(\theta) \tag{A.33}$$

$$\dot{\mu} = -[v_q(q(\theta), \theta)f(\theta) + \gamma(\theta)(\theta - c'(q)) + \Gamma(\theta)] \tag{A.34}$$

$$\lambda(\theta) = -\dot{\Lambda}(\theta) \geq 0, \quad \lambda(\theta)D(\theta) = 0 \tag{A.35}$$

In the fully revealing region where $q(\theta) = q_f(\theta)$, we have $\Gamma(\theta) = -v_q(q_f(\theta), \theta)f(\theta)$, as in the deterministic case.

Sufficiency/Concavity

By Kamien and Schwartz (1971), the necessary conditions are sufficient if the maximized Hamiltonian

$$\bar{H}(q, U, D, \gamma, \mu, \Gamma, \lambda, \Lambda) \equiv \max_{\nu, w} H(q, U, D, \nu, w, \gamma, \mu, \Gamma, \lambda, \Lambda)$$

is concave in (q, U, D) for given $(\gamma, \mu, \Gamma, \lambda, \Lambda)$, which requires $v_{qq}(q, \theta)f(\theta) - \gamma c''(q) \leq 0$. Concavity is satisfied if $\Gamma + \kappa F$ is increasing.

Proof of Proposition 3. Use the same multipliers as in the deterministic ratings (where $D(\theta) \equiv 0$), it suffices to check $\lambda(\theta) \geq 0$, the complementary-slackness condition for $D(\theta) \geq 0$ (MPS). Because $\lambda(\theta) = -\Lambda'(\theta)$ and $\Lambda(\theta) = -[\theta\Gamma(\theta)]'/f(\theta)$, $\lambda(\theta) \geq 0$ is satisfied if $\Lambda(\theta)$ is decreasing.

If f is increasing, then condition (S) is satisfied at θ_0 such that $\theta_1(\theta_0) = \bar{\theta}$. Because $\Gamma(\theta) = -A(\theta_0)$, $\Lambda(\theta) = -[\theta\Gamma(\theta)]'/f(\theta) = A(\theta_0)/f(\theta)$ is decreasing in θ.

If f is decreasing and $\underline{\theta} = 0$, then condition (C) is satisfied at $\theta_0 = 0$. Because $\Gamma(\theta) = -f(\theta)$ and $\theta f'(\theta)/f(\theta)$ is decreasing, $\Lambda(\theta) = -[\theta\Gamma(\theta)]'/f(\theta) = 1 + \theta f'(\theta)/f(\theta)$ is also decreasing in θ. □

Proof of Proposition 4. Sufficiency is shown in the proof of Proposition 3. In the fully revealing region, because $q(\theta) = q_f(\theta)$, by the Pontryagin maximum principle,

$$-\Lambda(\theta) = [\theta\Gamma(\theta)]'/f(\theta) = -(1 + \theta f'(\theta)/f(\theta))$$

and $\lambda(\theta) = -\Gamma'(\theta) \geq 0$ (the complementary-slackness condition for $D(\theta) \geq 0$ (MPS)) are also necessary for the optimality of the rating scheme. □

Appendix B Beyond Lower Censorship

B.1 Optimal Deterministic Ratings

In this section, I characterize sufficient conditions for the deterministic rating schemes that are not necessarily lower censorship to be optimal. As before, it is without loss to focus on the quality scheme it induces, which consists of pooling and fully revealing intervals and at most countably many jump discontinuities (see Lemma F.1). Therefore, given a quality scheme $q(\theta)$, I label the exclusion interval as $[\underline{\theta}, \theta_0]$, and other pooling and fully revealing intervals as $[\theta_0, \theta_1], \ldots, [\theta_{k-1}, \theta_k]$, where $\theta_0 < \theta_1 < \ldots < \theta_k = \bar{\theta}$ and $k \geq 1$.[33] As a convention, denote $\theta_{-1} = \underline{\theta}$. Define $q_j = q(\theta_j+)$ for all $j \geq 0$ and $q_{-1} = 0$. Thus, given a quality scheme $q(\theta)$, q_j is uniquely determined by θ_j, and $(q_{-1}, q_0, q_1, \ldots, q_{k-1})$ is an increasing sequence.

For any two adjacent pooling intervals $[\theta_{j-1}, \theta_j]$ (on which $q(\theta) = q_{j-1}$) and $[\theta_j, \theta_{j+1}]$ (on which $q(\theta) = q_j$), Lemma F.1 implies $q_{j-1} - c(q_{j-1})/\theta_j = q_j - c(q_j)/\theta_j$ at the jump θ_j. Thus, each jump θ_j determines a minimum standard $q_j > 0$.

Moreover, if a pooling interval is adjacent to a fully revealing interval, $q(\theta)$ must be continuous at the boundary of the pooling interval, i.e., $q_j = q_f(\theta_j)$ on the pooling

[33]The labeling of $(\theta_1, \ldots, \theta_k)$ is possible because $q(\theta)$ has at most countably many jumps.

interval.

Example (Lower censorship)**.** Lower censorship is a special case of $k \leq 2$. When $k = 2$, $[\underline{\theta}, \theta_0]$ and $[\theta_0, \theta_1]$ are the pooling intervals, and $[\theta_1, \bar{\theta}]$ is the fully revealing interval; q_0 is the only minimum standard.

Example (Two thresholds)**.** Assume $c(q) = q^2/2$ and $\Theta = [0, 5]$. Then, an incentive-compatible quality scheme can have two jumps (see Figure B.1), so the rating scheme has two thresholds $q_0 = 2$ and $q_1 = 4$.

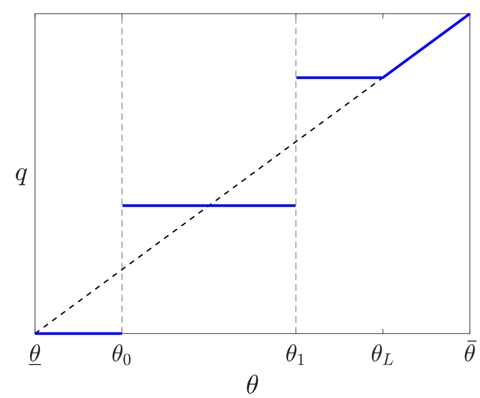


Figure B.1: $q(\theta)$ with two jumps induced by two thresholds

In general, the optimal control method can still be applied to solve for the optimal deterministic rating. Under Condition LD, define the characteristic function by

$$r_j(\theta) = (\beta(\theta) - \alpha\theta)f(\theta) - \alpha(F(\theta) - F(\theta_j)) \tag{B.1}$$

Define $R_j(\theta) = \int_{\underline{\theta}}^{\theta} r_j(\tilde{\theta})\, \mathrm{d}\tilde{\theta}$, $L_j(\theta|\theta_j) = \frac{R(\theta_c(\theta_j)) - R(\theta_j)}{\theta_c(\theta_j) - \theta_j}$, and $A_j = L_j(\theta_c(\theta_j)|\theta_j)$ for all $j \geq 0$.

I state the following sufficient conditions on the pooling and fully revealing intervals for the optimal deterministic rating scheme, as extensions of conditions (S) and (C).

Condition (S-j). On any two adjacent pooling intervals $[\theta_{j-1}, \theta_j]$ (where $q(\theta) = q_{j-1}$) and $[\theta_j, \theta_{j+1}]$ (where $q(\theta) = q_j$),

$$\int_{\theta_j}^{\theta} r_j(\tilde{\theta}) d\tilde{\theta} \geq A_j \cdot (\theta - \theta_j) \text{ for all } \theta \in [c'(q_{j-1}), c'(q_j)],$$

with equality if $\theta \in \{c'(q_{j-1}), c'(q_j)\} \cap (\underline{\theta}, \bar{\theta})$.[34]

[34] Recall the convention that $q_{-1} = 0$ and $\theta_{-1} = \underline{\theta}$

Condition (C-j). On any fully revealing interval $[\theta_j, \theta_{j+1}]$, $r_j(\theta)$ is decreasing in θ.

I propose the following sufficient conditions for the optimal deterministic rating scheme, extending conditions (S) and (C).

Proposition B.1. *A rating scheme is optimal among deterministic ratings if the induced quality scheme $q(\theta)$ satisfies conditions (S-j) and (C-j) hold on all pooling and fully revealing intervals, respectively, for some $\theta_0 < \theta_1 < \ldots < \theta_k = \bar{\theta}$.*

Analogous to the case of lower censorship, the sufficient conditions (S-j) and (C-j) are related to the modes of the density $r(\theta)$.

Corollary B.1.1. *If $r(\theta)$ has $n \geq 1$ modes, the optimal deterministic rating scheme has at most n thresholds.*[35] *If the smallest mode is in the interior of $[\underline{\theta}, \bar{\theta}]$, the optimal deterministic rating scheme has a minimum standard below which a "fail" signal is disclosed.*

B.2 Optimal General Ratings

Define

$$N_1(\theta) = \left(\frac{v_{qq}(q_f(\theta), \theta)}{c''(q_f(\theta))} + v_{q\theta}(q_f(\theta), \theta) \right) \theta + v_q(q_f(\theta), \theta) \left(1 + \frac{\theta f'(\theta)}{f(\theta)} \right). \tag{B.2}$$

Example (Linear Delegation)**.** Under Condition LD, $N_1(\theta) = (\beta'(\theta) - \alpha)\theta + (\beta(\theta) - \alpha\theta)[1 + \theta f'(\theta)/f(\theta)]$. When $v(q, \theta) = q$, $N_1(\theta) = 1 + \theta f'(\theta)/f(\theta)$.

Condition (D). $N_1(\theta)$ is decreasing in θ.

Lemma B.1. *If the optimal deterministic rating scheme fully reveals $\theta \in \Theta_f$, then the optimal rating scheme also fully reveals $\theta \in \Theta_f$ if and only if $N_1(\theta)$ is decreasing on Θ_f.*

Because it provides a necessary and sufficient condition, the lemma also implies that if the optimal deterministic rating scheme has a fully revealing region where $N_1(\theta)$ is not decreasing, then there exists a stochastic rating scheme strictly that improves upon it (see Proposition 4).

In the two adjacent pooling regions $[\theta_{j-1}, \theta_j]$ and $[\theta_j, \theta_{j+1}]$, the following condition needs to hold.

Condition (P). $N_2(\theta) = A_j/f(\theta) + \kappa\theta + \kappa(F(\theta) - F(\theta_j))/f(\theta)$ is decreasing in θ.

[35] When f is constant in some regions, there are potentially many optimal deterministic rating schemes (or $q(\theta)$), and I consider the one with the fewest thresholds (or jumps).

In particular, for lower censorship or pass/fail tests, $\theta_j = \theta_0$ and $A_j = A(\theta_0)$ as defined in equation (8).

The following proposition provides sufficient conditions for the optimal rating scheme to be deterministic.

Proposition B.2. *A rating scheme is optimal if the induced quality scheme $q(\theta)$ satisfies conditions in Proposition B.1 and Conditions (D) and (P) on the fully revealing and pooling regions, respectively.*

The following proposition, which follows immediately from Lemma B.1, provides sufficient conditions for the stochastic ratings to strictly improve upon the optimal deterministic ratings.

Proposition B.3. *The principal strictly benefits from stochastic rating schemes if the quality scheme induced by the optimal deterministic rating scheme has a fully revealing region in which Condition (D) does not hold.*

Appendix C Ability Signaling

C.1 Setup and Preliminaries

In this section, I consider the alternative case where the market only values the agent's exogenous ability, θ, à la Spence's (1973) signaling model.[36] In this case, the interim wage is $\hat{w}(q) = \mathbf{E}_{s\sim\pi(q)}[\mathbf{E}[\theta|s]]$ because $\omega(s) = \mathbf{E}[\theta|s]$. As before, denote $w(\theta) = \hat{w}(q(\theta))$.

The lemmas for the equivalence to reduced-form direct mechanism and incentive compatibility still hold. The theorem below provides a necessary and sufficient condition for feasibility.

Theorem C.1. *An incentive-compatible direct mechanism $(q(\theta), w(\theta))$ is feasible if and only $w(\theta)$ is a mean-preserving spread of θ in the quantile space, that is,*

(i) $\int_{\underline{\theta}}^{\theta} w(\theta')\,\mathrm{d}F(\theta') \geq \int_{\underline{\theta}}^{\theta} \theta'\,\mathrm{d}F(\theta')$ *for all* $\theta \in [\underline{\theta}, \bar{\theta}]$ *(MPS'),*

(ii) $\int_{\underline{\theta}}^{\bar{\theta}} w(\theta)\,\mathrm{d}F(\theta) = \int_{\underline{\theta}}^{\bar{\theta}} \theta\,\mathrm{d}F(\theta)$ *(BP').*

The proof is à la Kleiner, Moldovanu, and Strack (2021, Theorem 3, Border's theorem). Under deterministic rating schemes, $w(\theta)$ can only be an extreme point the

[36] In the employer example, it is similar to Holmström's (1999) career concern model, except that agents know their abilities.

mean-preserving spread of θ in the quantile space, which is referred to as a "truthful filter" in Rayo (2013).

Because the type θ is exogenous, the rating design problem is simpler than the case where the market values the endogenous quality. On the technical side, because (MPS') and (BP') do not involve the state variable $q(\theta)$, the Hamiltonian becomes simpler as it does not involve pure state constraint. Hence, in this section, I look for the optimal general (possibly stochastic) ratings directly, without having to start by restricting to deterministic ratings.

C.2 Optimal Rating Design

Because the test is costless and always gives a result, taking the test is a strictly dominant strategy for every agent (except the lowest type $\underline{\theta}$ who can be indifferent), even if he invests no effort (i.e., $c(q,\theta)=0$). Therefore, every agent participates in the test, even if he invests no effort, in contrast to the productive investment case. Consequently, $w_{\varnothing}=\underline{\theta}$.

Lemma C.1. *In any equilibrium, if an agent does not take the test, he must be the lowest type $\theta=\underline{\theta}$ who chooses $\underline{q}$ such that $c(\underline{q},\underline{\theta})=0$, and the market offers him $w_{\varnothing}=\underline{\theta}$.*

The principal's problem is

$$\max_{q(\theta),w(\theta)} \int_{\underline{\theta}}^{\bar{\theta}} v(q(\theta),\theta)\,\mathrm{d}F(\theta) \tag{C.1}$$

subject to (MPS'), (BP'), and

$$\theta w(\theta) - c(q(\theta)) = \int_{\underline{\theta}}^{\theta} w(x)\,\mathrm{d}x + \underline{U}, \qquad \text{(IC-Env)} \tag{C.2}$$

$$q(\theta) \text{ increasing.} \qquad \text{(IC-Mon)} \tag{C.3}$$

$$\theta w(\theta) - c(q(\theta)) \geq \underline{\theta}\cdot\theta, \qquad \text{(IR)} \tag{C.4}$$

Say a rating induces *full separation* if $w(\theta)=\theta$.[37] Define $q_f(\theta)$ as the quality scheme

[37]Cf. the fully revealing test in previous sections that induces $\hat{w}(q)=q$ $(w(\theta)=q(\theta))$ when the market values quality.

under full separation, which is characterized by

$$\hat{w}(q_f(\theta)) \equiv w(\theta) = \theta, \tag{BP}$$

$$q_f(\theta) = \arg\max_q \{\theta \hat{w}(q) - c(q)\} \iff \hat{w}'(q_f(\theta)) = c'(q_f(\theta))/\theta, \tag{FOC}$$

$$\underline{\theta} - c(q_f(\underline{\theta}))/\underline{\theta} = 0. \tag{IR}$$

The first two conditions imply

$$c'(q_f(\theta)) \cdot q_f'(\theta) = \theta, \tag{C.5}$$

which, along with the initial condition in (IR), determines $q_f(\theta)$.

I also maintain Assumption 1 (downward bias) that $y_q(q_f(\theta), \theta) \geq 0$. Denote

$$J(\theta|q_f) = \frac{y_q(q_f(\theta), \theta)}{c'(q_f(\theta))}\theta - \frac{\int_\theta^{\bar{\theta}} y_q(q_f(x), x)/c'(q_f(x)) \, \mathrm{d}F(x)}{f(\theta)} \tag{C.6}$$

In the linear delegation case where $v(q, \theta) = \beta(\theta)q - \alpha c(q) + d(\theta)$, the expression simplifies to

$$J(\theta|q_f) = \frac{\beta(\theta)}{c'(q_f(\theta))}\theta - \frac{\int_\theta^{\bar{\theta}} \beta(x)/c'(q_f(x)) \, \mathrm{d}F(x)}{f(\theta)} - \alpha \left(\theta - \frac{1 - F(\theta)}{f(\theta)} \right) \tag{C.7}$$

If the principal maximizes expected quality—i.e., $v(q, \theta) = q$, then $J(\theta|q_f) = \frac{\theta}{c'(q_f(\theta))} - \frac{\int_\theta^{\bar{\theta}} 1/c'(q_f(x)) \, \mathrm{d}F(x)}{f(\theta)}$.

Proposition C.2. *The optimal rating scheme induces full separation (i.e., $q^*(\theta) = q_f(\theta)$) if and only if $J(\theta|q_f)$ is increasing in θ.*

Proof sketch. Rewrite the constraints and apply the optimal control methods to the principal's maximization problem. See Appendix C.3 for details. □

Remark 5. For $v(q, \theta) = q$, the result is consistent with Rayo (2013) (which assumes $c(q) = q$) and Zubrickas (2015, Propositions 2 and 3) but does not restrict attention to deterministic ratings.

The necessary and sufficient condition regarding $J(\theta|q_f)$ is reminiscent of that for the optimality of winner-take-all contests in Zhang (2024). Indeed, effort maximization in the ability signaling model is similar to that in contests.

Proposition C.2 provides a regularity condition that is necessary and sufficient for full separation to be optimal. In particular, if $v(q, \theta) = q$ and $c(q) = q$, full separation is optimal if and only if $J(\theta) \equiv \theta - \frac{1-F(\theta)}{f(\theta)}$ is increasing.[38]

Example C.1. Assume $v(q,\theta) = q$, $c(q) = q$ (as in Rayo (2013)), and $\underline{\theta} = 0$. Then $\hat{w}(q) = \sqrt{2q}$ and $q_f(\theta) = \theta^2/2$. The optimal rating induces full separation $q^*(\theta) = \theta^2/2$ if and only if $J(\theta|q_f) = \theta - \frac{1-F(\theta)}{f(\theta)}$ is increasing.

Example C.2. Assume $v(q,\theta) = q$, $c(q) = q^2/2$, and $\underline{\theta} = 0$. Then, $\hat{w}(q) = q$ and $q_f(\theta) = \theta$. The optimal rating induces full separation $q^*(\theta) = \theta$ if and only if $J(\theta|q_f) = 1 - \frac{\int_\theta^{\bar{\theta}} 1/x \, \mathrm{d}F(x)}{f(\theta)}$ is increasing.

The following corollary implies that in the quality maximization case, the optimal rating induces full separation at the top under some conditions.

Corollary C.2.1 (Cf. Zubrickas, 2015, Propositions 2)**.** *Assume* $v(q,\theta) = q$*. If* $c'(q_f(\theta))/\theta$ *is decreasing in* θ *(or equivalently,* $q_f(\theta)$ *is convex) on* $[\theta_1, \bar{\theta}]$ *for sufficiently large* $\theta_1 < \bar{\theta}$*, then the optimal rating induces full separation on* $[\theta_1, \bar{\theta}]$.

C.3 Proofs of Appendix C

Proof of Proposition C.1. Full separation leads to $w(\theta) = \theta$, while pooling on $[\theta_1, \theta_2]$ leads to $w(\theta) = \mathbf{E}[\theta \mid \theta \in [\theta_1, \theta_2]]$. In particular, total pooling leads to $w(\theta) = \mathbf{E}[\theta]$.

($\Longrightarrow$) follows from $\mathbf{E}[w(\theta) \mid \theta \geq \tau] \leq \mathbf{E}[\theta \mid \theta \geq \tau]$ for all $\tau \in [\underline{\theta}, \bar{\theta}]$ because switching to full separation reveals more information about high types. ($\Longleftarrow$) is by applying Choquet's Theorem to the extreme points of the MPS of θ in the quantile space (i.e., pooling or fully separating). □

Proof of Proposition C.2. I prove the proposition using optimal control method.

Setup of Hamiltonian

The setup of Hamiltonian is almost identical to Appendix A.3, except that the state equation of D is replaced by $\dot{D} = [w(\theta) - \theta]f(\theta)$ due to (MPS').[39]

Set up the Hamiltonian

$$\begin{aligned} H = v(q(\theta),\theta)f(\theta) + \gamma(\theta)[\theta w(\theta) - c(q(\theta)) - U(\theta)] + \lambda(\theta)D(\theta) \\ + \Lambda(\theta)[w(\theta) - \theta]f(\theta) + \Gamma(\theta)w(\theta) + \mu(\theta)\nu(\theta) \end{aligned} \tag{C.8}$$

[38]In the quality maximization case with linear cost, Kleiner et al. (2021, Proposition 2) implies that optimal rating scheme is always deterministic because the maximum of a linear function is always obtained at an extreme point.

[39]Consequently, $D \geq 0$ is no longer a pure state constraint, making the problem easier to solve.

where U, q, D are the state variables and w, ν are the control variables.

By the Pontryagin's maximum principle, the necessary conditions are

$$-\frac{\partial H}{\partial q} = -(v_q f - \gamma c'(q)) = \dot{\mu} \tag{C.9}$$

$$-\frac{\partial H}{\partial D} = -\lambda = \dot{\Lambda} \tag{C.10}$$

$$-\frac{\partial H}{\partial U} = \gamma = \dot{\Gamma} \tag{C.11}$$

$$\frac{\partial H}{\partial w} = \theta\gamma + \Lambda f + \Gamma = 0 \tag{C.12}$$

$$\frac{\partial H}{\partial \nu} = \mu \leq 0, \quad \mu(\theta) = 0 \text{ if } q \text{ is strictly increasing at } \theta \tag{C.13}$$

$$\lambda(\theta) \geq 0, \quad \lambda(\theta)D(\theta) = 0 \tag{C.14}$$

$$\Gamma(\underline{\theta}) \leq 0, \quad \Gamma(\underline{\theta})U(\underline{\theta}) = 0 \tag{C.15}$$

$$\mu(\underline{\theta}) \leq 0, \quad \mu(\underline{\theta})q(\underline{\theta}) = 0 \tag{C.16}$$

$$\Gamma(\bar{\theta}) = 0, \quad \mu(\bar{\theta}) = 0 \tag{C.17}$$

$$\Lambda(\bar{\theta}) \text{ no condition.} \tag{C.18}$$

Proposed Multipliers.

I focus on the full revelation region where $q(\theta) = q_f(\theta)$. Because $q(\theta) = q_f(\theta)$ ($\dot{q} \geq 0$ is not binding), we have $\gamma(\theta) = v_q(q_f(\theta), \theta)f(\theta)/c'(q_f(\theta))$ and thus

$$\Gamma(\theta) = -\int_{\theta}^{\bar{\theta}} \frac{v_q(q_f(x), x)}{c'(q_f(x))} f(x)\,\mathrm{d}x.$$

Hence,

$$-\Lambda(\theta) = -\frac{\theta\gamma(\theta) + \Gamma(\theta)}{f(\theta)} = \frac{v_q(q_f(\theta), \theta)\theta}{c'(q_f(\theta))} - \frac{\int_{\theta}^{\bar{\theta}} v_q(q_f(x), x) f(x)/c'(q_f(x))dx}{f(\theta)} \equiv J(\theta|q_f) \tag{C.19}$$

Therefore, the complementary-slackness condition for (MPS'), $\lambda(\theta) = -\Lambda'(\theta) \geq 0$, holds if and only if $J(\theta|q_f) = -\Lambda(\theta)$ is increasing in θ.

Sufficiency/Concavity.

Note that the Hamiltonian is concave (and hence the maximized Hamiltonian). In particular, it is concave in q because

$$v_{qq}(q_f(\theta),\theta) - \gamma(\theta)c''(q_f(\theta)) = v_{qq}(q_f(\theta),\theta) - v_q(q_f(\theta),\theta)\cdot c''(q_f(\theta))/c'(q_f(\theta)) \leq 0, \quad \text{(C.20)}$$

because $v_q(q_f(\theta),\theta) \geq 0$ and $v_{qq}(q_f(\theta),\theta) \leq 0$. It is also linear in (U, D) and control valuables (w,ν).

Hence, the condition that $J(\theta|q_f)$ is increasing is necessary and sufficient for $q_f(\theta)$ being the optimal solution. □

Proof of Corollary C.2.1. $q_f(\theta)$ is convex if and only if $\theta/c'(q_f(\theta))$ is increasing in θ. The second term of $J(\theta)$, $J_2(\theta) = -\frac{\int_\theta^{\bar\theta} 1/c'(q_f(x))\,\mathrm{d}F(x)}{f(\theta)}$, is increasing in θ for sufficiently large $\theta_1 < \bar\theta$ because $J_2'(\theta) = \frac{c'(q_f(\theta))f(\theta)+f'(\theta)\int_\theta^{\bar\theta} 1/c'(q_f(x))\,\mathrm{d}F(x)}{f(\theta)^2}$. □

Appendix D Comparison with Amador and Bagwell (2022)

Optimal deterministic rating design is equivalent to optimal deterministic delegation with an outside option (Amador and Bagwell, 2022, henceforth AB), where lower censorship corresponds threshold delegation, and pass/fail tests correspond to take-it-or-leave-it offers. Compared to AB, I obtain stronger results that provide necessary and sufficient conditions for threshold delegation (i.e., price-cap allocation) to be optimal, thereby allowing for the optimality of a bang-bang allocation where the firm either shuts down or always sets the price at the cap (which can also be implemented by take-it-or-leave it offers).

In this section, I compare my conditions with theirs by providing sufficient conditions for lower censorship to be optimal à la AB in my setting using their approach.[40]

Truncated Problem

AB first fix a cutoff θ_0 and look at the truncated problem for $\theta \geq \theta_0$. Define

$$G(\theta|\theta_0) = \frac{1}{\theta_1(\theta_0)-\theta}\int_\theta^{\theta_1(\theta_0)} v_q(\tilde\theta, q_i(\theta_0))f(\tilde\theta)d\tilde\theta + \kappa\frac{\theta_c(\theta_0)-\theta}{\theta_1(\theta_0)-\theta}(1-F(\theta)) - \kappa(1-F(\theta_0)), \quad \text{(D.1)}$$

[40]Alternatively, in Xiao (2023a), I provide necessary and sufficient conditions for price-cap allocations to be optimal in their setting using the same method.

where $\theta_c(\theta_0) = c'(q_i(\theta_0))$.

Their Proposition 1 proposes the following two conditions for threshold delegation (i.e., price-cap allocation) to be optimal in the truncated problem.

Condition (AB(i)). $G(\theta|\theta_0) \le G(\theta_0|\theta_0)$ for all $\theta \in [\theta_0, \theta_1(\theta_0)]$.

Condition (AB(ii)). $v_q(\theta, q_f(\theta))f(\theta) - \kappa F(\theta)$ is decreasing in θ on $(\theta_1(\theta_0), \bar{\theta}]$.

Observation D.1. *If* $r(\theta) = v_q(\theta, q_i(\theta_0))f(\theta) - \kappa(\theta - \theta_c(\theta_0))f(\theta) - \kappa(F(\theta) - F(\theta_0))$ *is decreasing on* $[\theta_0, \theta_1(\theta_0)]$ *(G'), then* $G(\theta|\theta_0)$ *is decreasing on* $\theta \in [\theta_0, \theta_1(\theta_0)]$*, and condition AB(i) holds.*

Condition AB(ii) is the same as condition (C). For Condition AB(i), recall that condition (S) can be decomposed into conditions (S1) and (S2) on the pooling regions and exclusion regions, respectively.

Condition (S1). $L(\theta|\theta_0) \ge L(\theta_c(\theta_0)|\theta_0) = A(\theta_0)$ for all $\theta \in (\theta_0, \theta_1(\theta_0)]$.

Condition (S2). $L(\theta|\theta_0) \le L(\theta_c(\theta_0)|\theta_0) = A(\theta_0)$ for all $\theta \in [0, \theta_0)$.

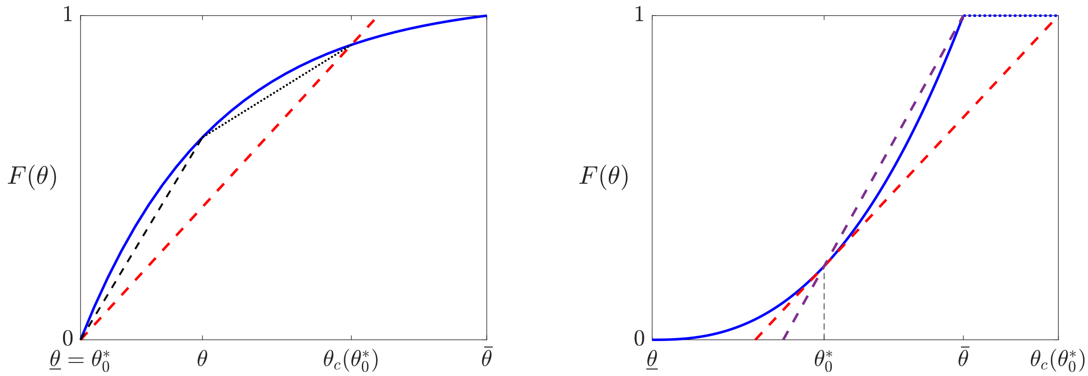
Condition (S2) has no counterpart in AB's conditions because they focus on the truncated problem for $\theta \ge \theta_0$. The following observation shows that (S1) is less restrictive than AB(i).

Observation D.2. *Condition AB(i) is equivalent to* $L(\theta|\theta_0) \ge L(\theta_1(\theta_0)|\theta_0)$ *for all* $\theta \in [\theta_0, \theta_1(\theta_0)]$.

Remark 6. Condition (S1) is less restrictive than AB(i) because $\theta_c(\theta_0) \ge \theta_1(\theta_0) = \min\{\theta_c(\theta_0), \bar{\theta}\}$. Consequently, condition AB(i) implies a fully revealing region by ruling out the possibility that $\theta_c(\theta_0) > \bar{\theta}$ (e.g., when $r(\theta)$ is increasing). Thus, pass/fail tests or bang-bang allocations are never optimal under condition AB(i).

For example, if $r(\theta) = f(\theta)$, Figure D.1 illustrates conditions AB(i) and (S1). In the left panel, the red dashed line represents $L(\theta_c(\theta_0)|\theta_0)$ and $L(\theta_1(\theta_0)|\theta_0) = G(\theta_0|\theta_0)$. They coincide because $\theta_c(\theta_0) \le \bar{\theta}$ (and hence $\theta_c(\theta_0) = \theta_1(\theta_0)$). For a fixed $\theta \in [\theta_0, \theta_1(\theta_0)]$, the black dashed line represents $L(\theta|\theta_0)$, while the black dotted line represents $G(\theta|\theta_0)$; the former has a higher slope than the red dashed line if and only if the latter has a lower slope than the red line. Thus, condition AB(i) and condition (S1) are equivalent if $\theta_c(\theta_0) \le \bar{\theta}$.

In the right panel, the purple dashed line represents $L(\theta_1(\theta_0)|\theta_0)$, while the red dashed line represents $L(\theta_c(\theta_0)|\theta_0)$. Contrary to the previous case, because $\theta_c(\theta_0) > \bar{\theta}$ (e.g., if f is increasing), f satisfies condition (S1) but violates condition AB(i).



(a) Satisfies both AB(i) and (S1)

(b) Satisfies (S1) but violates AB(i)

Figure D.1: Graphic Illustration of Conditions AB(i) vs. Condition (S1)

On the technical side, the differences between condition AB(i) and condition (S1) is because I propose a smaller multiplier A. The multiplier à la AB, denoted by A^{AB}, is given by

$$\begin{aligned} A^{\text{AB}} &\equiv \frac{1}{\theta_1(\theta_0) - \theta_0} \int_{\theta_0}^{\theta_1(\theta_0)} [v_q(q_i(\theta_0), \theta) f(\theta) - \kappa f(\theta)(\theta - \theta_c(\theta_0)) - \kappa(F(\theta) - F(\theta_0))] \,\mathrm{d}\theta \\ &= \frac{1}{\theta_1(\theta_0) - \theta_0} \left[\int_{\theta_0}^{\theta_1(\theta_0)} v_q(q_i(\theta_0), \theta) f(\theta) \,\mathrm{d}\theta - \kappa(\theta_1(\theta_0) - \theta_c(\theta_0))(1 - F(\theta_0)) \right] = G(\theta_0|\theta_0). \end{aligned}$$

By contrast, the multiplier A I propose is

$$A = \frac{1}{\theta_c(\theta_0) - \theta_0} \int_{\theta_0}^{\theta_1(\theta_0)} v_q(q_i(\theta_0), \theta) \,\mathrm{d}F(\theta) \leq A^{\text{AB}}, \tag{D.2}$$

where the equality holds if and only if $\theta_c(\theta_0) \leq \bar{\theta}$ (so that $\theta_1(\theta_0) = \theta_c(\theta_0)$). Consequently, their multiplier A^{AB} requires that a fully revealing region $[\theta_1(\theta_0), \bar{\theta}]$ must be nonempty.

Global Problem

Then, for global optimality, AB's Proposition 2 requires the two conditions in the truncated problem to hold *for all* $\theta_0 \in [\underline{\theta}, \bar{\theta})$. In principle, these conditions need not hold at exclusion levels θ_0 that are dominated (e.g., θ_0 close to $\bar{\theta}$). The following proposition shows that requiring them to hold for all $\theta_0 \in [\underline{\theta}, \bar{\theta})$ rules out the possibility that the optimal allocation has exclusion.[41]

[41]The optimal price-cap allocation in AB still has exclusion because they assume a fixed production cost.

Proposition D.1 (Amador and Bagwell (2022) (Propositions 1 and 2))**.** *If conditions AB(i) and AB(ii) hold for all $\theta_0 \in [\underline{\theta}, \bar{\theta})$, the optimal deterministic rating is lower censorship without exclusion.*

Proof. In the spirit of AB, fix $\theta_0 \in [\underline{\theta}, \bar{\theta})$ and look at the truncated problem for $\theta \geq \theta_0$. Because condition AB(i) implies condition (S1) with $\theta_c(\theta_0) \leq \bar{\theta}$, while condition AB(ii) is the same as condition (C), Proposition 2 implies the optimal quality scheme (in the truncated problem) is

$$q(\theta) = \begin{cases} q_i(\theta_0), & \text{if } \theta \in [\theta_0, \theta_1(\theta_0)) \\ q_f(\theta), & \text{if } \theta \in [\theta_1(\theta_0), \bar{\theta}]. \end{cases} \tag{D.3}$$

Because conditions AB(i) and AB(ii) hold for all $\theta_0 \in [\underline{\theta}, \bar{\theta})$, they hold at $\theta_0 = \underline{\theta}$ in particular, so the optimal deterministic rating scheme is lower censorship with cutoff $\theta_0^* = \underline{\theta}$. □

Appendix E Optimal Ratings with Transfers

E.1 Transfers Contingent on the Rating Result or Quality

In this subsection, I consider a transfer scheme $T(s) \in \mathbb{R}$ contingent on the rating result $s \in S$ from the agent to the principal. Alternatively, under the interpretation that π is a disclosure policy (i.e., the principal can observe the agent's quality q), I also consider a certification fee contingent on the agent's quality q. The following lemma shows that in either case, the transfer scheme can provide incentives in place of the rating scheme.[42]

Lemma E.1. *Two pairs of test-fee schedules $\{\pi_1, T_1(s)\}$ and $\{\pi_2, T_2(s)\}$ always induce the same quality scheme if $\hat{w}_1(q) - \hat{T}_1(q) = \hat{w}_2(q) - \hat{T}_2(q)$, where $\hat{w}_i(q) \equiv \mathbf{E}_{s\sim\pi_i(q)}[\mathbf{E}[\tilde{q}|s]]$ and $\hat{T}_i(q) \equiv \mathbf{E}_{s\sim\pi(q)}[T_i(s)]$.*

Thus, with result-contingent fees, it is without loss to focus on a fully revealing (i.e., the most informative) test $\bar{\pi}$ such that $\hat{w}(q) = q$ and vary the transfer scheme $T(s)$. Alternatively, if the principal can observe quality and design quality-contingent fees $P(q)$ directly (i.e., π is a disclosure policy), a wide range of disclosure-fee schedules can implement the same quality scheme, as long as the transfer scheme is calibrated accordingly to provide the same incentives. In other words, the design of the rating scheme becomes irrelevant.

[42]The perfect substitutability is also noted by Albano and Lizzeri (2001) (with moral hazard) and Pollrich and Strausz (2024) (with pure adverse selection).

By the similar argument as the revelation principle (and the taxation principle), it is equivalent to focus on a feasible direct mechanism $(q(\theta), w(\theta), t(\theta))$, where $w(\theta) = \mathbf{E}_{s\sim\pi(q(\theta))}[\mathbf{E}[q|s]]$ is the interim wage and $t(\theta) = \mathbf{E}_{s\sim\pi(q(\theta))}[T(s)]$ is the interim transfer.

Assume the principal's objective is $v(q(\theta), \theta) + (1 + \lambda)t(\theta)$, where λ captures the weight of transfers $t(\theta)$ relative to $v(q, \theta)$ in her objective. The principal's problem becomes

$$\max_{q,w,t} \int_{\underline{\theta}}^{\bar{\theta}} v(q(\theta), \theta) + (1 + \lambda)t(\theta) \, \mathrm{d}F(\theta) \tag{E.1}$$

subject to (IC), (IR), (MPS), and (BP). With interim transfers, the agent's utility in (IC) and (IR) becomes $U(\hat{\theta}|\theta) = w(\hat{\theta}) - c(q(\hat{\theta}), \theta) - t(\hat{\theta})$, so the envelope condition is given by

$$w(\theta) - c(q(\theta), \theta) - t(\theta) = -\int_{\underline{\theta}}^{\theta} c_\theta(q(x), x)dx. \tag{IC-Env'}$$

As soon as the envelope equation and Bayesian plausibility $\mathbf{E}[w(\theta)] = \mathbf{E}[q(\theta)]$ is substituted into the principal's objective, the problem reduces to a classical mechanism design problem with transfers (e.g., Baron and Myerson, 1982; Laffont and Tirole, 1993).

Proposition E.1. *Assume F satisfies IFR. The optimal quality scheme $q^*(\theta)$ is given by*

$$c_q(q^*(\theta), \theta) = 1 + \frac{1}{1 + \lambda} v_q(q^*(\theta), \theta) + \frac{1 - F(\theta)}{f(\theta)} c_{\theta q}(q^*(\theta), \theta) \tag{E.2}$$

which can be implemented by a fully revealing test $\bar{\pi}(q) = q$ and a result-contingent (or quality-contingent) certification fee

$$T^*(s) = s - \int_{q^*(\underline{\theta})}^{s} c_q(u, q^{*-1}(u)) \, \mathrm{d}u - c(q^*(\underline{\theta}), \underline{\theta}), \tag{E.3}$$

The optimal fee scheme $T^*(s)$ leaves no information rent for the lowest type $\underline{\theta}$. The certification fee also increases as the agent's quality increases, appropriating the agent's gain from quality investment while also leaving information rent for agents.

The optimal quality scheme $q^*(\theta)$ is distorted downward from the first-best quality $q^{FB}(\theta)$, which satisfies $c_q(q^{FB}(\theta), \theta) = 1 + v_q(q^{FB}(\theta), \theta)$. In the extreme case where $\lambda \to \infty$ (that is, the principal is a monopoly certifier), we have $c_q(q^*(\theta), \theta) = 1 + \frac{1-F(\theta)}{f(\theta)} c_{\theta q}(q^*(\theta), \theta)$ (see Albano and Lizzeri, 2001).

E.2 Constant Testing Fees

Because the design of the rating scheme becomes irrelevant when result-contingent (or qualit-contingent) transfers are allowed, I now consider a *constant* testing fee. In reality, laws and regulations usually require certification fees to be upfront, flat fees. Moreover, if the principal can tamper with the rating, then the restriction to a constant testing fee is required for incentive compatibility of principals.

When the principal can design stochastic ratings, Albano and Lizzeri (2001) show that the revenue-maximizing rating scheme stochastic: it reveals quality with some probability and outputs the same signal for every participant otherwise (see also Saeedi and Shourideh (2020); Xiao (2023b)). More generally, for a principal (e.g., regulatory certifier) who maximizes a weighted sum of the certification fee, Xiao (2025, Chapter 3) shows that if the agent payoff and cares more about the former, a noisy test remains optimal, and the agent always underinvests in quality compared to the first-best level. Additionally, pass/fail tests are revenue-maximizing among *deterministic* ratings.

Appendix F Deferred Results and Proofs

F.1 Results and Proofs of Section 4

Lemma F.1. *An incentive-compatible quality scheme $q(\theta)$ consists of pooling intervals (where $q(\theta)$ is constant) and full revealing intervals (where $q(\theta) = q_f(\theta)$) with at most countably many jump discontinuities.*

At each discontinuity $\hat{\theta} \in [\underline{\theta}, \bar{\theta}]$, the following conditions must hold.

1. $q(\hat{\theta}-) - c(q(\hat{\theta}-))/\hat{\theta} = q(\hat{\theta}+) - c(q(\hat{\theta}+))/\hat{\theta}$,

2. $q(\theta) = q(\hat{\theta}-)$ *for* $\theta \in [q_f^{-1}(q(\hat{\theta}-)), \hat{\theta})$ *and* $q(\theta) = q(\hat{\theta}+)$ *for* $\theta \in (\hat{\theta}, q_f^{-1}(q(\hat{\theta}+))]$,

3. $q(\hat{\theta}) \in \{q(\hat{\theta}-), q(\hat{\theta}+)\}$,

where $q_f^{-1}(\cdot) = \max\{\min\{c'(\cdot), \bar{\theta}\}, \underline{\theta}\}$*, and* $q(\hat{\theta}-)$ *and* $q(\hat{\theta}+)$ *denote the left and right limit of* $q(\theta)$ *at* $\hat{\theta}$*.*

Proof of Lemma F.1. Because $q(\theta)$ is increasing, it has at most countably many jump discontinuities and is differentiable almost everywhere. Assume without loss that $q(\theta)$ is right-continuous so that the right-derivative $q'(\theta+) \equiv \lim_{h \to 0^+} \frac{q(\theta+h)-q(\theta)}{h}$ always exists. Then, (IC) implies $(c'(q(\theta)) - \theta)q'(\theta) = 0$, so either $q(\theta) = q_f(\theta)$ or $q'(\theta) = 0$.

At each discontinuity, conditions 1 and 3 follow from the convex and thus absolute continuity of $U(\theta)$. Condition 2 follows from the first part ($q'(\theta) = 0$) and continuity of $U(\theta)$ (which determines the interval endpoints). □

Claim 2 (Optimality of "no rent at the bottom")**.** *If $v_q(q_i(\theta), \theta) \geq 0$ for all $\theta \in [\underline{\theta}, \theta_c(\underline{\theta})]$, then the optimal cutoff $\theta_0^* \geq \underline{\theta}$. Thus, the lowest type has no information rent (i.e., $\underline{U} = 0$).*

Intuitively, if $v_q(q_i(\theta), \theta) \geq 0$ for all $\theta \in [\underline{\theta}, \theta_c(\underline{\theta})]$, the principal can always benefit from a higher minimum standard that push the lowest type $\underline{\theta}$ to the boundary of the (IR) condition without increasing exclusion.

Proof of Claim 2. If $v_q(q_i(\theta), \theta) \geq 0$ for all $\theta \in [\underline{\theta}, \theta_c(\underline{\theta})]$, then $v_q(q_i(\underline{\theta}), \theta) \geq 0$ for all $\theta \in [\underline{\theta}, \theta_c(\underline{\theta})]$, so $A(\theta) \geq 0$ for all $\theta \leq \underline{\theta}$. Thus, for all $\theta_0 < \underline{\theta}$, $f(\theta_0) = 0$ implies $V'(\theta_0) = A(\theta_0)q_i(\theta_0) - v(q_i(\theta_0), \theta_0)f(\theta_0) = A(\theta_0)q_i(\theta_0) \geq 0$. Hence, $\theta_0^* \geq \theta_0$. □

Claim 3 (Optimality of no exclusion)**.** *If $f(\theta)$ is decreasing and $v_{q\theta}(q, \theta) \leq -v_{qq}(q, \theta)/c''(q)$ for all $q \in Q$ and $\theta \in [\underline{\theta}, \bar{\theta}]$, then no exclusion is optimal (i.e., $\theta_0^* \leq \underline{\theta}$).*

Proof of Claim 3. Recall that $\kappa = \inf_{q,\theta}\{-v_{qq}/c''(q)\}$. If $v_{q\theta}(q, \theta) \leq \kappa$, then $d(q, \theta) = v(q, \theta) - \kappa(\theta q - c(q))$ satisfies $d_{qq} \leq 0$ and $d_{q\theta} \leq 0$. Therefore,

$$\int_{\theta_0}^{\theta_1(\theta_0)} v_q(q_i(\theta_0), \theta)\,\mathrm{d}\theta = \int_{\theta_0}^{\theta_1(\theta_0)} d_q(q_i(\theta_0), \theta)\,\mathrm{d}\theta + \int_{\theta_0}^{\theta_1(\theta_0)} \kappa(\theta - c'(q))\,\mathrm{d}\theta$$
$$\leq d_q(q_i(\theta_0), \theta_0)(\theta_1(\theta_0) - \theta_0) \leq \frac{d(q_i(\theta_0), \theta_0)}{q_i(\theta_0)}(\theta_1(\theta_0) - \theta_0) = \frac{v(q_i(\theta_0), \theta_0)}{q_i(\theta_0)}(\theta_1(\theta_0) - \theta_0)$$

Then, because f is decreasing,

$$\int_{\theta_0}^{\theta_1(\theta_0)} v_q(q_i(\theta_0), \theta) f(\theta)\,\mathrm{d}\theta \leq f(\theta_0)\int_{\theta_0}^{\theta_1(\theta_0)} v_q(q_i(\theta_0), \theta)\,\mathrm{d}\theta \leq \frac{v(q_i(\theta_0), \theta_0)}{q_i(\theta_0)} f(\theta_0)(\theta_1(\theta_0) - \theta_0)$$

for all $\theta_0 \in (\underline{\theta}, \bar{\theta})$. Finally, we have

$$V'(\theta_0) \leq \left(\int_{\theta_0}^{\theta_1(\theta_0)} \frac{v_q(q_i(\theta_0), \theta)}{\theta_1(\theta_0) - \theta_0} f(\theta)\,\mathrm{d}\theta - \frac{v(q_i(\theta_0), \theta_0)}{q_i(\theta_0)} f(\theta_0)\right) q_i(\theta_0) \leq 0$$

because $c'(q_i(\theta_0)) \geq \theta_1(\theta_0) > \theta_0$. □

Lemma (A.3)**.** *If f is unimodal on $[\underline{\theta}, \bar{\theta}]$, then it is quasi-unimodal. If f is increasing on $[\underline{\theta}, \bar{\theta}]$, then it is quasi-increasing. If f is decreasing on $[\underline{\theta}, \bar{\theta}]$, then it is quasi-decreasing; the converse is true if $\underline{\theta} = 0$. If $\bar{\theta} \leq \theta_c(\underline{\theta})$, then every unimodal $f(\theta)$ is quasi-increasing.*

Proof of Lemma A.3. (i) For simplicity, assume $r(\theta)$ is strictly unimodal with mode $\theta_m \in (\underline{\theta}, \bar{\theta})$, so that $R(\theta)$ is convex-concave on $[\underline{\theta}, \bar{\theta}]$ with a reflection point θ_m (note that R is decreasing for all $\theta \geq \bar{\theta}$). Therefore, it satisfies conditions (S) and (C) at a unique $\theta_0 \in (\theta_c^{-1}(\theta_m), \theta_m)$ such that $\theta_c(\theta_0) \geq \theta_m$, which is straightforward from Figure 3. The formal proof is tedious and deferred to the Appendix F.

To see this formally, denote $\phi(\theta) = R(\theta_c(\theta_0)) - R(\theta_0) - r(\theta_0)(\theta_c(\theta_0) - \theta_0)$. By the mean value theorem, $\frac{R(\theta_c(\theta_0))-R(\theta_0)}{\theta_c(\theta_0)-\theta_0} = r(\xi)$ for some $\xi \in (\theta_0, \theta_c(\theta_0))$. If $\theta_c(\theta_0) < \theta_m$, then r is strictly increasing, so $r(\xi) > r(\theta_0)$. If $\theta_0 > \theta_m$, then r is strictly decreasing, so $r(\xi) < r(\theta_0)$. Hence, $\phi(\theta_0) > 0$ if $\theta_0 < \theta_c^{-1}(\theta_m)$ and $\phi(\theta_0) < 0$ if $\theta_0 > \theta_m$, so there exists some $\theta_0 \in (\theta_c^{-1}(\theta_m), \theta_m)$ such that $\phi(\theta_0) = 0$; moreover, $\phi(\theta_0) = 0$ only if $\theta_0 \in (\theta_c^{-1}(\theta_m), \theta_m)$.

To establish uniqueness of θ_0, note that $\phi'(\theta) = \theta_c'(\theta_0)(r(\theta_c(\theta_0)) - r(\theta_0)) - (\theta_c(\theta_0) - \theta_0)r'(\theta_0) < 0$ for all $\theta_0 \in (\theta_c^{-1}(\theta_m), \theta_m)$ because $r'(\theta_0) > 0$ and $r(\theta_c(\theta_0)) < r(\xi) = r(\theta_0)$ for some $\xi \in (\theta_m, \theta_c(\theta_0))$. (We know that $r(\xi) = r(\theta_0)$ for some $\xi \in (\theta_0, \theta_c(\theta_0))$ by the mean value theorem, and that $\xi > \theta_m$ because r is strictly increasing $\theta < \theta_m$.)

(ii) Analogously, an increasing $r(\theta)$ satisfies condition (S) at some $\theta_0 \in [\theta_c^{-1}(\bar{\theta}), \bar{\theta})$ such that $\theta_c(\theta_0) \geq \bar{\theta}$ and thus satisfies condition (C) vacuously. It can be viewed as a special case of the unimodal $r(\theta)$ with $\theta_m = \bar{\theta}$.

(iii) A decreasing $r(\theta)$ satisfies conditions (S) and (C) at $\theta_0 = \underline{\theta}$. If $\underline{\theta} = 0$, then $\theta_c(\underline{\theta}) = 0$, so a quasi-decreasing function is decreasing by condition (C). □

F.2 Proofs of Appendix B

Proof Sketch of Proposition B.1. The proof is similar to that of Proposition 1 by applying the arguments recursively to the pooling and fully revealing intervals.

Corollary B.1.1 follows from the fact that n-mode function can satisfy condition (C-j) and (S-j) for at most n points $(\theta_1, \ldots, \theta_n)$, which is by similar arguments to the proof of Lemma A.3. □

Proof of Lemma B.1. (Necessity.) On Θ_f, fully revealing ($q = q_f(\theta)$) implies

$$\Gamma(\theta) = -v_q(q_f(\theta), \theta) f(\theta),$$

$$\Lambda(\theta) = -[\theta\Gamma(\theta)]'/f(\theta) = \frac{v_{qq}(q_f(\theta), \theta)}{c''(q_f(\theta))}\theta + v_{q\theta}(q_f(\theta), \theta)\theta + v_q(q_f(\theta), \theta)[1 + \theta f'(\theta)/f(\theta)],$$

which must be decreasing because the complementary-slackness on $D \geq 0$ (MPS) condition implies that the Lagrangian multiplier $\lambda(\theta) = -\Lambda'(\theta) \geq 0$.

(Sufficiency.) Because the optimal deterministic rating fully reveals Θ_f, condition (C) is satisfied on Θ_f (i.e., $\Gamma + \kappa F$ is increasing), and the Hamiltonian is concave. Therefore, the necessary conditions for optimality are also sufficient. □

Proof of Proposition B.2. Then, Lemma B.1 implies the sufficiency of condition (D) for the fully revealing region.

In the pooling regions, the multiplier for the optimal deterministic rating is $\Gamma(\theta) = -A(\theta_j) - \kappa(F(\theta) - F(\theta_j))$, so

$$\Lambda(\theta) = -\frac{[\theta\Gamma(\theta)]'}{f(\theta)} = \frac{A(\theta_j)}{f(\theta)} + \kappa\theta + \kappa\frac{F(\theta) - F(\theta_j)}{f(\theta)},$$

which must be decreasing because the Lagrangian multiplier $\lambda(\theta) = -\Lambda'(\theta) \geq 0$. The conditions for optimality are also sufficient because conditions in (B.1) guarantee the concavity of the Hamiltonian (i.e., $\Gamma + \kappa F$ is increasing). □

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