Incentivizing Agents through Ratings*

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Abstract

I study the optimal design of ratings to motivate agent investment in quality when transfers are unavailable. The principal designs a rating scheme that maps the agent's quality to a (possibly stochastic) score. The agent has private information about his ability, which determines his cost of investment, and chooses the quality level. The market observes the score and offers a wage equal to the agent's expected quality. For example, a school incentivizes learning through a grading policy that discloses the student's quality to the job market.

I reduce the principal's problem to the design of an interim wage function of quality. When restricted to deterministic ratings, I provide necessary and sufficient conditions for the optimality of simple pass/fail tests and lower censorship. In particular, when the principal's objective is expected quality, pass/fail tests are optimal if agents' abilities are concentrated towards the top of the distribution, while pass/lower censorship is optimal if abilities are concentrated towards the mode. The results generalize existing results in optimal delegation with voluntary participation, as pass/fail tests (lower censorship) correspond to take-it-or-leave-it offers (threshold delegation). Additionally, I provide sufficient conditions for deterministic ratings to remain optimal when stochastic ratings are allowed. For quality maximization, pass/fail tests remain optimal if the ability distribution becomes increasingly more concentrated towards the top.

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1 Introduction

In many economic applications, a principal seeks to motivate agents' performance or investment in quality, but monetary transfers between them are prohibited or limited. In these situations, the principal can instead incentivize agents through a rating scheme (or disclosure policy) that reveals information about their performance or quality to the market. When the market rewards agents based on this information, ratings can provide reputational incentives for agents.

For example, consider a school in which students invest productive efforts to improve their quality (i.e., human capital). Suppose the school wants to incentivize student investment to achieve better placement outcomes, maximize tuition fees, or encourage human capital formation. To maximize its objective, the school designs a grading rule that provides information about students' endogenous quality to the job market. Similarly, regulatory certifiers who care about consumer welfare use quality certifications to motivate firm investment in product quality.[1] Employers (e.g., pre-doc positions) may pay a fixed wage to employees and induce effort through ratings (e.g., recommendation letters) that disclose information about their performance and inherent abilities to future employers.[2] In these examples, the market pays the agent the expected value for his endogenous quality (or inherent ability) conditional on the rating result. By contrast, transfers between the principal and agent contingent on the rating result are infeasible in practice or prohibited by law—for instance, certification fees are usually required to be upfront flat fees.[3,4]

Various rating schemes are used in these environments to motivate agents. A frequently observed scheme is *pass/fail* tests. Licensing exams, such as bar examinations, are often pass/fail. Pass/fail is also ubiquitous in product certifications, such as UL Certifications and ISO Certifications. Another prevalent form of test is *lower censorship*, which reveals quality if and only if it exceeds a minimum standard. For example, some schools release precise scores (or class ranks) above a failing grade. In product certifications, lower censorship is commonly known as quality assurance, which censors low-quality

[1] Regulatory or NGO certifiers care about overall product quality because of consumer welfare (see Zapechelnyuk, 2020; Bizzotto and Harstad, 2023) or spillovers of product quality. Journal editors can also be viewed as certifiers who care about the product (i.e., paper) quality. By contrast, monopoly certifiers only care about certification fees.

[2] Future employers may also value past performance *per se* because of learning by doing.

[3] Starting from 2008, Wall Street's three major credit rating agencies will receive upfront fees for reviewing mortgage-backed securities regardless if they are hired to assign a rating (Graybow and Siew, 2008).

[4] Such restrictions on transfers can also arise from economic constraints, such as incentive compatibility or budget balance. For instance, if the certifier can tamper with the rating, then the restriction to a flat certification fee is required for incentive compatibility of certifiers.

products that do not meet the standard and prevent them from being sold on the market. Yet another form is coarse letter grades that consist of multiple minimum standards. For instance, students who meet the lower standard but not the higher one get a "low-pass" grade. Alternatively, ratings may involve randomness, such as random inspection or disclosure of product quality. For example, the certifier may use an algorithm that determines the probability of checking or disclosing the product quality.

In this paper, I study the optimal design of rating schemes to motivate agent investment in quality when transfers are unavailable. Instead, the principal designs a rating scheme (à la Blackwell) that maps the agent's quality to a (possibly stochastic) score. The agent has private information about his ability, which determines his cost of investment, and chooses the quality level. The market observes the score and offers a wage equal to the agent's expected quality.

The problem is more challenging than it might appear. At first glance, full revelation (or full disclosure) of quality might seem to be the most motivating scheme because any marginal investment in quality will be revealed to the market. This is true for a utilitarian principal who has the same preference as the agent. However, when the principal wants to incentivize quality investment, a minimum standard can provide stronger incentives for some agents, as they need to invest more in quality to separate themselves from the low levels that fail to meet the standard. Therefore, tests with (one or more) minimum standards, such as pass/fail and coarse grading, can be optimal.[5] Alternatively, stochastic rating schemes can potentially provide stronger incentives for some types than deterministic rating schemes. As will be seen in the paper, I solve the problem using optimal control methods.

To characterize the optimal rating scheme, I reduce the rating design problem to the equivalent problem of designing an incentive-compatible direct mechanism that consists of a quality function and an *interim* wage function. The interim wage function maps the agent's type to the expected wage he receives from the market in equilibrium. Because the agent's wage is offered by the market based on his score and the rating scheme, the mechanism design problem is subject to a feasibility constraint that the interim wage must be a mean-preserving spread of the quality in the quantile space.

My first set of results concerns the optimal *deterministic* rating schemes. A deterministic rating scheme either fully reveals quality or pools some qualities to the same score. In the latter case, among the qualities that are pooled to the same score, only the lowest

[5]This argument does not hinge on cognitive or technological costs (or constraints) of precise information, which are not considered in this paper. These costs and constraints will make pass/fail tests and coarse grading more likely to be optimal.

one will be chosen by the agent.[6] Thus, the interim wage always equals the quality, as the market can perfectly infer the agent's quality from his score. Using optimal control methods, I provide sufficient conditions for the optimal deterministic rating scheme to be lower censorship or a simple pass/fail test. The conditions are also necessary if the principal's marginal payoff from the agent's quality is linear in (a transformation of) the quality.[7] In particular, when the principal maximizes expected quality, lower censorship is optimal if agents' abilities are concentrated around the mode of the distribution (e.g., unimodal density). If abilities are concentrated towards the top (e.g., increasing density), a pass/fail test maximizes the expected quality. Otherwise, if abilities are concentrated towards the bottom (e.g., decreasing density), lower censorship with a minimum standard that every type will meet in equilibrium maximizes the expected quality. Intuitively, when there are more high types, it is more profitable to set a high minimum standard to induce higher investment in quality from high types, even if it excludes some low types. Specifically, the optimal minimum standard is such that passing requires even the highest type to invest more than he would under full revelation. On the other hand, when there are more low types, excluding them to incentivize high types becomes unprofitable, so the optimal minimum standard will allow the lowest type to barely reach it in the equilibrium.

Beyond lower censorship and pass/fail tests, I solve for the optimal deterministic ratings for general distributions and preferences. The solution extends the conditions for lower censorship to accommodate multiple (alternating) pooling and full revelation regions. For example, if the ability density is bimodal,[8] the quality-maximizing deterministic rating can take the form of high-pass/low-pass/fail.

My results also have implications for optimal delegation because the deterministic rating design problem is equivalent to optimal deterministic delegation with voluntary participation (see also Zapechelnyuk, 2020). In the delegation problem (à la Holmstrom (1984)), the principal determines a set of permissible actions and delegates the agent to choose one from the set (or the outside option). Similarly, in the deterministic rating design problem, the principal effectively designs a set of undominated qualities for the agent to choose.[9] Thus, pass/fail tests correspond to take-it-or-leave-it offers, while lower censorship corresponds to threshold delegation. My results generalize existing

[6]This is no longer the case if agent investment determines quality stochastically.

[7]This is referred to as "linear delegation" in the delegation literature.

[8]See, e.g., Carrell et al. (2013) for empirical evidence of bimodal ability distribution in United States Air Force Academy squadrons.

[9]To see this, when multiple qualities are pooled to the same score, the lowest quality among them will strictly dominate others.

results in the literature on delegation with voluntary participation (Amador and Bagwell, 2022; Kartik, Kleiner and Van Weelden, 2021) by providing necessary and sufficient conditions for threshold delegation and take-it-or-leave-it offers, while allowing for general principal preferences that can depend on the agent's type (i.e., state-dependent). Analogously, other deterministic rating schemes also have their counterparts in the delegation problem.

My second set of results considers settings where stochastic rating schemes are allowed. A natural question is whether stochastic ratings can improve on deterministic ratings. First, I provide sufficient conditions under which deterministic rating schemes (including lower censorship and pass/fail tests) remain optimal. In the quality maximization case, pass/fail tests remain optimal if the ability density is increasing. Second, I also identify conditions under which stochastic ratings can strictly improve on deterministic ratings. For example, a noisy test that partially pools low quality with high quality enables the principal to increase the incentives for low types at the cost of incentives for high types, which can potentially increase the overall expected quality. This is true when the ability density has a heavy tail—that is, there are a few very high ability agents.

As an extension, I consider the ability signaling case where the market values the agent's exogenous ability instead of endogenous quality. In other words, the agent's effort is signaling rather than productive. The rating design problem can also be reduced to a mechanism design problem subject to a feasibility constraint that the interim wage must be a mean-preserving spread of the *ability* in the quantile space. Because ability is exogenous, the problem is simpler. With linear costs, the quality-maximizing rating is always deterministic and induces full separation if and only if the ability distribution is regular in the sense of Myerson (1981).

Methodologically, the paper uses recent advances in optimal control methods to account for jumps in the optimal quality scheme (Hellwig, 2008, 2010; Clarke, 2013). Because there are no transfers, the Myersonian approach is not applicable, and neither are the standard optimal control methods (e.g., Guesnerie and Laffont (1984)) because they require the quality scheme (i.e., state variable) to be absolutely continuous.[10] Thus, I use the maximum principle formulated by Hellwig (2008, 2010) to account for the monotonicity of the quality function without assuming its absolute continuity. Moreover, because of voluntary participation, the optimal quality scheme induced by lower censorship and pass/fail has a jump at the (endogenous) cutoff type. I use the switching condition in the

[10]With type-contingent transfers, it can be shown that the optimal scheme has no jumps (see, e.g., Mussa and Rosen (1978) and Kamien and Schwartz (2012, Section 18)), so one can assume absolute continuity and use its derivative as a control variable.

hybrid maximum principles (Clarke, 2013; Bryson and Ho, 1975) to account for the jump at the cutoff.

The paper makes three main contributions to the literature. First, I provide a unified framework to study the optimal rating scheme to motivate agents, which allows the principal to have a state-dependent preference and to design stochastic rating schemes, and in which the agent's effort can be either productive or signaling. If the principal (e.g., school) internalizes part of the agent's (e.g., student's) cost, her preference will depend on the agent's type. Moreover, the principal may have a state-dependent bliss point and a quadratic-loss payoff, as is common in the literature (e.g., Alonso and Matouschek (2008)). Thus, a state-dependent preference is an important consideration. Second, my results for optimal deterministic ratings have implications for optimal delegation with voluntary participation. I generalize existing results by providing necessary and sufficient conditions for threshold delegation and take-it-or-leave-it offers, allowing for the optimality of bang-bang allocations (Cf. Amador and Bagwell, 2022), state-dependent principal preferences (Cf. Kartik, Kleiner and Van Weelden, 2021), and nonlinear delegation (Cf. Kolotilin and Zapechelnyuk, 2019). Through the equivalence established by Kolotilin and Zapechelnyuk (2019) between delegation problems and Bayesian persuasion problems, the results also contribute to the persuasion literature, especially in the nonlinear case. Third, I contribute to the growing literature that uses the characterization of the posterior mean *conditional* on the agent's type (see Saeedi and Shourideh, 2020; Doval and Smolin, 2022, 2024). I develop the *interim* approach by Saeedi and Shourideh (2020), especially in the ability signaling case, that reduces the rating design problem to the optimization over interim wage functions rather than Blackwell experiments themselves.

Literature Review. This paper provides a framework that incorporates two strands of literature on the optimal rating design to motivate agents when a competitive market pays the expected value. A strand of literature assumes the market values the agent's *endogenous* quality or effort (Albano and Lizzeri, 2001; Saeedi and Shourideh, 2020, 2023; Zapechelnyuk, 2020; Rodina and Farragut, 2020; Boleslavsky and Kim, 2021). Zapechelnyuk (2020) studies the optimal *deterministic* quality certification to incentivize sellers' investment in product quality and characterize sufficient conditions for lower censorship and pass/fail certifications. His conditions for pass/fail require small differences in agents' abilities.[11] Compared to the literature, my conditions for lower censorship

[11] Rodina and Farragut (2020) also show the effort-maximizing deterministic grading rule (i) has a failing grade combined with full revelation if the distribution is sufficiently concave, (ii) has two or three letter grades if the distribution is sufficiently convex, and (iii) has at most three letter grades combined with full revelation if the distribution is sufficiently single-peaked.

and pass/fail are less restrictive. I also allow for state-dependent preferences (e.g., the principal partially internalizes the agent's cost) and stochastic rating schemes.[12]

Another strand of literature assumes the market values the *exogenous* abilities à la Spence's (1973) signaling or Holmström's (1999) career concern model (Dewatripont, Jewitt and Tirole, 1999; Rayo, 2013; Zubrickas, 2015; Rodina, 2020; Hörner and Lambert, 2021; Onuchic and Ray, 2023). Rayo (2013) and Zubrickas (2015) characterize the conditions under which the effort-maximizing deterministic rating scheme induces full separation or pooling of agents.[13,14] My results for ability signaling (Section 6) generalize them by allowing for stochastic ratings and general objective functions.[15] More recently, Onuchic and Ray (2023) assume the market values both inherent ability and learning effort and characterize conditions under which the optimal *deterministic* grading scheme to incentivize learning (and signal student abilities) is lower censorship or full pooling.[16] My paper complements theirs by focusing on common priors while allowing for stochastic rating schemes and assuming convex effort costs so that returns to quality investment alone (without signaling) can still motivate some investment.[17]

As mentioned above, this paper is also related to the delegation literature with limited transfers (Holmstrom, 1984; Alonso and Matouschek, 2008) and voluntary participation (Amador and Bagwell, 2022; Kartik, Kleiner and Van Weelden, 2021). Amador and Bagwell

[12]Some papers consider stochastic rating schemes with a constant (i.e., noncontingent) transfer (i.e., certification fee). Albano and Lizzeri (2001) shows the rating scheme that maximizes certification fees is stochastic—it reveals quality with some probability and outputs an (almost) uninformative signal otherwise. Saeedi and Shourideh (2020) extend it to the case where the principal maximizes a weighted average of the agents' payoff using the interim approach. Saeedi and Shourideh (2023) assume agent investment increases quality stochastically. Additionally, Boleslavsky and Kim (2021) consider stochastic rating schemes without transfers but assume agent investment improves the *distribution* of his quality.

[13]In particular, pooling (or coarse grading) can be optimal when the ability distribution is not regular in the sense of Myerson (1981). Similarly, when in agents care about their relative rankings, Moldovanu et al. (2007) find coarse partitions can be effort-maximizing if the ability distribution is not regular—in particular, two categories are optimal if the distribution is sufficiently concave. Dubey and Geanakoplos (2010) find the optimal grading scheme to motivate students is coarse if their abilities are "disparate"—i.e., a higher-ability student can achieve a higher score even when he shirks.

[14]Some papers also find pass/fail tests or coarse grading to be optimal for reasons other than incentivizing agents. Harbaugh and Rasmusen (2018) find coarse grading can be more informative to the receiver because of increased participation. DeMarzo et al. (2019) show there is no benefit for a more precise test than pass/fail when the agent can choose whether to disclose the test result.

[15]In a dynamic career concerns model, Hörner and Lambert (2021) show the effort-maximizing rating is a linear function of past observations.

[16]This is an application of their more general model, which studies the optimal monotone categorization of (exogenous) ability to maximize the principal's expected payment of the market when the market (receiver) values ability and has different priors from the principal (sender).

[17]They assume linear effort costs so that returns to quality investment alone does not motivate any investment (which is a corner solution). If students have common priors, the school effectively maximizes students' effort (i.e., endogenous quality) while internalizing a constant fraction of their costs, which is encompassed in state-dependent preferences.

(2022) study the problem of regulating a monopolist without transfers and characterize sufficient conditions for threshold delegation (i.e., price-cap regulation) to be optimal. Compared to them, my results correspond to *necessary* and sufficient conditions of price-cap regulation and take-it-or-leave-it offers, thereby allowing for the optimality of a bang-bang allocation where the firm either sets the price at the cap or shuts down.[18] In contrast to expertise-based delegation, Kartik et al. (2021) study delegation in veto-bargaining with an outside option. They assume the principal has a specific state-independent preference (with a constant bliss point) and identify the necessary and sufficient conditions for the optimality of interval delegation, including full delegation and no compromise (i.e., take-it-or-leave-it), among possibly stochastic delegation mechanisms. In comparison, I allow for general state-*dependent* preferences, and stochastic rating schemes in my setting are not equivalent to stochastic delegation mechanisms, despite their equivalence in the deterministic case.[19]

The method I use in characterizing optimal deterministic ratings develops the Lagrangian methods in the delegation literature advanced by Amador, Werning and Angeletos (2006) (see also Amador and Bagwell, 2013, 2022) to account for jumps in the optimal allocation (particularly due to the participation constraint) using optimal control tools (Bryson and Ho, 1975; Hellwig, 2008, 2010; Clarke, 2013).[20] The method tackles the delegation problem directly without invoking the equivalence to persuasion.[21] Furthermore, my method allows for nonlinear delegation (i.e., the principal's marginal payoff is nonlinear in the agent's action; see Section 4.2.4) and can be extended to stochastic ratings using the interim wage function and the feasibility condition.[22]

Outline. The rest of the paper is organized as follows. In Section 2, I introduce the model and assumptions. In Section 3, I reduce the principal's problem to the design of a direct mechanism subject to a feasibility constraint. Section 4 studies optimal deterministic ratings. In Section 5, I consider optimal general (potentially stochastic) ratings and also extend the model to allow for a constant testing fee.[23] Section 6 explores the ability signaling case where the market values the agent's exogenous ability.

[18] Under their sufficient conditions for price-cap regulation, the bang-bang allocation is never optimal. See also Halac and Yared (2022) for the optimality of bang-bang solutions.

[19] Rappoport (2022) studies a delegation problem when agents have career concerns, which is related to the ability signaling case but different in that the agent prefers to take higher actions than the principal.

[20] See also Xiao (2023a).

[21] This is also different from the method used in Kleiner et al. (2021, Section 4.3) for linear delegation.

[22] As noted above, despite the equivalence to delegation under deterministic ratings, the rating design problem with stochastic ratings in Section 5 is *not* equivalent to stochastic delegation.

[23] In Appendix E, I show that ratings are irrelevant if type-contingent transfers are allowed, and I extend the model to allow for a constant (i.e., noncontingent) transfer if the agent takes the test.

2 The Model

2.1 Setup

An agent (he) has a private type θ, which has a continuous distribution F with support $\Theta = [\underline{\theta}, \bar{\theta}]$ and continuous density $f(\theta) > 0$. He can choose a quality level $q \in Q \equiv [0, q_{\max}]$ at cost $c(q, \theta)$, which is twice continuously differentiable and satisfies $c_q(q, \theta) > 0$, $c_\theta(q, \theta) < 0$, $c_{qq}(q, \theta) > 0$, $c_{q\theta}(q, \theta) < 0$ for all $q > 0$ and $\theta \in \Theta$. Assume $c(0, \theta) = c_q(0, \theta) = 0$ for all $\theta \in \Theta$.

The principal has a utility function given by $v(q, \theta)$, which is twice continuously differentiable and satisfies $v_{qq}(q, \theta) \leq 0$, $v(0, \theta) = 0$, and $v_q(0, \theta) > 0$ for all $q \in Q$ and $\theta \in \Theta$ (see Assumptions 1 for another assumption on $v_q(q, \theta)$). The principal does not observe θ. It does not matter whether the principal observes q as long as the rating takes it as input. If she observes q, the rating scheme is a disclosure policy that garbles the quality; otherwise, it is a test that inputs the quality and outputs a score. Assume there are no transfers between the principal and agent. Instead, the principal can design a rating (Blackwell experiment) $\pi\colon Q \to \Delta(S)$ to reveal information about the agent's quality q and hence type θ to the market and provide reputational incentives.

The market values the agent's quality q. Assume the market is competitive has a payoff $-(\omega - q)^2$ when she pays a wage ω to an agent of quality q. If the agent takes the test, the market observes the score $s \in S$ and offers a wage $\omega(s) = \mathbf{E}[q|s]$. Thus, the agent's interim wage, as a function of his quality choice q, is $\hat{w}(q) = \mathbf{E}_{s\sim\pi(q)}[\omega(s)]$.[24] Hence, if the agent takes the test, he chooses $q \in Q$ to maximize his payoff $u(q, \theta) = \hat{w}(q) - c(q, \theta)$. Otherwise, he can also choose not to take the test, in which case the market observes a null signal $s = \varnothing$ and offers him a wage $\omega(\varnothing)$.

See Section 2.3 for a discussion of the model assumptions.

Timing. The timing of the game is as follows.

1. Agents privately learn their types $\theta \in [\underline{\theta}, \bar{\theta}]$.

2. The principal commits to a rating scheme $\pi\colon Q \to \Delta(S)$, which is publicly observed.

3. Agents choose quality $q \in Q$ and whether to take the test. If an agent takes the test, the market observes a signal s drawn from $\pi(q)$. Otherwise, the market observes a null signal $s = \varnothing$.

[24]The interim wage is the interim posterior mean of quality. To see this, denote by $\mu_s \in \Delta(Q)$ the posterior belief induced by s; then $\omega(s) = \mathbf{E}_{\mu_s}[q]$ is the posterior mean and $\hat{w}(q) = \mathbf{E}_{s\sim\pi(q)}[\mathbf{E}_{\mu_s}[\tilde{q}]]$ is the *interim* posterior mean.

4. The market updates her belief of the agent's quality to μ_s using Bayes' rule and sets the wage equal to the expected value, i.e., $\omega(s) = \mathbf{E}[q|s] \equiv \mathbf{E}_{\mu_s}[q]$.

Solution Concept. I study Sequential Equilibrium. The results are the same if I use weak Perfect Bayesian Equilibrium (wPBE) as the solution concept. Because the quality investment is costly, in any equilibrium, if an agent does not take the test, he must choose $q = 0$. Thus, the market must believe he has chosen $q = 0$ and offer $\omega(\varnothing) = 0$ accordingly.

Lemma 1. *In any equilibrium, if an agent does not take the test on the equilibrium path, then he chooses $q = 0$, and the market will offer him $\omega(\varnothing) = 0$.*

This lemma implies that the agent's outside option is zero. Moreover, because $v_q(0, \theta) > 0$, the optimal rating scheme will induce $\hat{w}(0) = 0$, so that an agent who chooses $q = 0$ and takes the test will also gets zero payoff.

2.2 Notation and Assumptions

Assume the cost is multiplicatively separable, that is, slightly abusing the notation, $c(q, \theta) = c(q)/\theta$, where $c'(q) > 0$, $c''(q) \geq 0$, and $c(0) = c'(0) = 0$.

Define the agent's quality choice under full revelation as

$$q_f(\theta) = \arg\max_{q \in Q} q - c(q)/\theta \iff c'(q_f(\theta)) = \theta. \tag{1}$$

In other words, $q_f(\theta)$ is the quality level at which the marginal cost $c'(q) = \theta$, so $q_f = c'^{-1}|_\Theta \colon \Theta \to Q$.

Define the principal-optimal quality as $q_e(\theta) = \arg\max_{q \in Q} v(q, \theta)$. I assume that the agent has a downward bias, i.e., $q_f(\theta) \leq q_e(\theta)$.[25]

Assumption 1 (Downward bias)**.** $v_q(q_f(\theta), \theta) \geq 0$ for all $\theta \in [\underline{\theta}, \bar{\theta}]$.

Define the quality choice $q_i(\theta)$ indifferent to the outside option (zero) by

$$\theta q_i(\theta) - c(q_i(\theta)) = 0, \quad \text{and } q_i(\theta) \geq q_f(\theta). \tag{2}$$

In other words, $q_i(\theta)$ is the quality level at which the average cost $\mathrm{AC}(q) \equiv c(q)/q = \theta$, so $q_i = \mathrm{AC}^{-1}|_\Theta \colon \Theta \to Q$. By the convexity of $c(q)$ and $c(0) = 0$, a unique $q_i(\theta) \geq q_f(\theta)$ exists

[25] When $q_f(\theta) > q_e(\theta)$, the principal will use a noisy rating (i.e., a garbling of the fully revealing test such that $w'(\theta) < q'(\theta)$) to implement $q < q_f(\theta)$.

for all $\theta \in [\underline{\theta}, \bar{\theta}]$ with equality if and only if $\theta = 0$. In particular, $q_i(0) = q_f(0) = 0$. Note that it is without loss to assume $q_{\max} = q_i(\bar{\theta})$ because no types will choose a higher quality.

Define $\theta_c(\theta) = c'(q_i(\theta))$ as the type that would choose $q_i(\theta)$ under full revelation. Note that $\theta_c\colon [\underline{\theta}, \bar{\theta}] \to [\underline{\theta}, +\infty)$, so it is possible that $\theta_c(\theta) > \bar{\theta}$, in which case $\theta_c(\theta)$ is a "hypothetical type" that would choose $q_i(\theta)$ under full revelation. Whenever $\theta_c(\theta) \leq \bar{\theta}$, $q_f(\theta_c(\theta)) = q_i(\theta)$.

Example 2.1. For quadratic cost $c(q) = q^2/2$, we have $q_f(\theta) = \theta$, $q_i(\theta) = 2\theta$, and $\theta_c(\theta) = 2\theta$.

Example 2.2. Assume $v(q, \theta) = \beta(\theta)q - \alpha c(q)$ with $\beta(\theta) > 0$ and $\alpha \geq 0$. Assumption 1 requires $\beta(\theta) \geq \alpha\theta$. The quality maximization case $v(q, \theta) = q$ satisfies the assumption.

2.3 Discussion of Assumptions

The market values quality. I assume the market values the (endogenous) quality q and not directly on the (exogenous) ability θ to shut down signaling.[26] This captures the cases in (i) the school example when learning is productive rather than signaling, (ii) the product certification example when the consumer values the product quality, and (iii) the employer-employee example when the performance accumulates human capital (learning by doing) or skills valued by the future employer. In Section 6, I assume the market values the ability θ à la Spence's (1973) signaling model.[27] In other words, the agent's effort is signaling rather than productive. That case will be simpler because abilities are exogenous.

The misalignment of incentives. I assume every agent has a downward bias, i.e., $v_q(q_f(\theta), \theta) \geq 0$ for all θ. This is the case, for example, when the principal internalizes only a fraction $\alpha \in [0, 1]$ of the agent's cost, i.e., $v(q, \theta) = q - \alpha c(q, \theta)$, a special case being $v(q, \theta) = q$, where the principal's objective is expected quality. Below, I provide several strands of micro-foundation of this misalignment.

First, a rational principal might not care about the costs. For example, the employer only wants to induce higher effort or output q from employees. A bureaucrat may just maximize output q and not care as much about the costs. The principal may also have an ideal q higher than the agent's desired q, independent of the agents types (Kartik et al., 2021).

[26]The market value can be easily generalized to a function of q if the cost function is adjusted accordingly.

[27]Rayo (2013) and Zubrickas (2015) study this case with $v(q, \theta) = q$ and restrict attention to deterministic ratings, while I allow for general $v(q, \theta)$ and stochastic ratings.

Second, it can be due to a benevolent principal because social cost of investment in quality can be lower than the private cost due to network effects. Equivalently, agent investment can also have some social benefits or spillovers in addition to the private benefits. For example, the school can be solely concerned with average human capital accumulation $\mathbf{E}[q]$ (Zubrickas, 2015) or put more weight on q relative to the agent's cost than the agent does. Similarly, The certifier can also be concerned with the overall quality of the products in the market. When the (government or NGO) certifier maximizes a weighted sum of firms' profit and average quality (Bizzotto and Harstad, 2023), the principal's objective is $\mathbf{E}[\alpha q(\theta) + (1-\alpha)U(\theta)] = \mathbf{E}[q(\theta) - \alpha c(q(\theta), \theta)]$.[28]

Third, it can arise from the agent's behavioral bias. The student may overestimate his cost or disutility of learning, possibly due to procrastination (DellaVigna and Malmendier, 2004).

Lastly, this misalignment can result from more complicated models. For example, the school maximizes the student's placement outcome (i.e., expected wage) for reputational reasons, which is equal to the expected quality (i.e., $\mathbf{E}[w(\theta)] = \mathbf{E}[q(\theta)]$) in this model. In Onuchic and Ray (2023, Section 4), the school maximizes tuition fee charged at the ex-ante stage (before students learn their types), which is equal to the students' expected payoff $\mathbf{E}[q(\theta) - \alpha c(q(\theta), \theta)]$, as their parents who pay the tuition internalize only a fraction α of the cost.[29] Moreover, in Zapechelnyuk (2020), the regulatory certifier maximizes consumer surplus, which is equivalent to maximizing average quality $\mathbf{E}[q(\theta)]$ under some assumptions.[30]

The role of (no) transfers. I rule out transfers to focus on the role of ratings in providing incentives. With contingent transfers $t(\theta)$, the design of ratings no longer matters because $t(\theta)$ can provide the same incentives in place of $w(\theta)$ through redistribution (see Appendix E.1).[31] I also consider a *constant* (i.e., noncontingent) testing fee that affects the optimal rating design through the agent's participation constraint for deterministic ratings (Appendix E.2) and stochastic ratings (Section 5.4 and Appendix E.3), as well as

[28]To see this, $\mathbf{E}[\alpha q(\theta) + (1-\alpha)U(\theta)] = \mathbf{E}[\alpha q(\theta) + (1-\alpha)(w(\theta) - c(q(\theta), \theta))] = \mathbf{E}[q(\theta) - \alpha c(q(\theta), \theta)]$.

[29]This is distinct from a constant testing fee mentioned above at the interim stage (after agents observe their type) because agents have no information rent.

[30]Specifically, he assumes the consumer has an outside option that follows the power distribution and buy the product if the price is lower than the outside option, which micro-founds a downward sloping demand. Given the demand function, the firm (i.e., agent) with private information about its cost (i.e., type) sets a price to maximize its profit. In this case, consumer surplus equals the expected quality after some transformation.

[31]On the technical side, with contingent transfers $t(\theta)$ as a control variable, there are no longer pure state constraints, and the maximized Hamiltonian is strictly concave in q, which implies the state variable $q(\theta)$ has no jumps (Kamien and Schwartz, 2012).

for the alternative case à la Spence (1973) (Section 6.3).

Multiplicatively separable cost. This assumption rules out another commonly used cost function: $c(q,\theta) = c(q-\theta)$, that is, a type-$\theta$ agent can invest effort $e \geq 0$ at the cost $c(e)$ to achieve quality $q = \theta + e$,[32] which I explore in Appendix F. When the market values quality q (type θ), this effort is productive (signaling). By contrast, in the main specification $c(q,\theta) = c(q)/\theta$, the effort is implicit, which can be interpreted as identical to the quality q. In Appendix F, I show that the results can extend to cost function $c(q,\theta) = c(q-\theta)$ and more general cost functions.

3 Preliminaries

3.1 Revelation Principle and Feasibility

The rating scheme can be viewed as an implementation of a direct mechanism, which does not require the agent's quality q to be observable by the principal, as long as it is taken as input by the rating scheme π. To see this, when the principal commits to a rating scheme $\pi\colon Q \to \Delta(S)$, the agent obtains a signal s drawn from $\pi(q)$ when he chooses quality $q \in Q$ and participates in the rating scheme. Alternatively, the principal can offer a *direct* mechanism $(q(\theta), s(\theta))$. If the agent accepts this mechanism, he reports his type θ, and is then required to choose quality level $q(\theta)$ and receives a (possibly stochastic) score $s(\theta)$ drawn from $\pi(q(\theta))$. By the revelation principle and the taxation principle, these two mechanisms are equivalent—i.e., choosing q is equivalent to reporting θ.[33]

Formally, say a quality function $q\colon \theta \to Q$ is *implementable* by a rating scheme $\pi\colon Q \to \Delta(S)$ if π and q satisfy the incentive compatibility constraint

$$\hat{w}(q) - c(q,\theta) \geq \hat{w}(q') - c(q',\theta) \quad \text{for all } \theta \in [\underline{\theta}, \bar{\theta}] \text{ and } q' \in Q, \tag{3}$$

where $\hat{w}(q) = \mathbf{E}_{s\sim\pi(q)}[\omega(s)]$ is the interim wage induced by π and $\omega(s) = \mathbf{E}[q|s]$.

Instead of optimizing over Blackwell experiments $\pi\colon Q \to \Delta(S)$, it is easier to work with the interim wage $\hat{w}\colon Q \to \mathbb{R}_+$ induced by π, which can be viewed as a "quasi-transfer" scheme offered by the market equal to its expectation of q based on the score and the rating scheme. Say a quality function $q(\theta)$ is *implementable* by $\hat{w}(q)$ if they satisfy the

[32]This cost function has been used in, among others, Laffont and Tirole (1993); Augias and Perez-Richet (2023); Perez-Richet and Skreta (2022).

[33]See also Fudenberg and Tirole (1991, Remark on pp. 257).

incentive compatibility constraint (3). Hence, $q(\theta)$ is implementable by $\pi(q)$ if and only if it is implementable by $\hat{w}(q) = \mathbf{E}_{s\sim\pi(q)}[\mathbf{E}[\tilde{q}|s]]$.

Because the signal $s(\theta)$ only affects the agent through the expected wage, I shall focus on a direct mechanism $(q(\theta), w(\theta))$ consisting of a quality function $q(\theta)$ and the interim wage function $w(\theta) = \hat{w}(q(\theta))$.

Definition 1. A direct mechanism $(q(\theta), w(\theta))$ consists of a quality function $q\colon [\underline{\theta}, \bar{\theta}] \to Q$ and an interim wage function $w \equiv \hat{w} \circ q\colon [\underline{\theta}, \bar{\theta}] \to \mathbb{R}_+$.

Unlike a transfer function between the principal and agent, the interim wage function $w(\theta)$ is offered by the market based on the agent's score and the rating scheme. This calls for the following definition of feasibility.

Definition 2. A direct mechanism $(q(\theta), w(\theta))$ is *feasible* if there exists a rating scheme $\pi\colon Q \to \Delta(S)$ such that $w(\theta) = \hat{w}(q(\theta)) \equiv \mathbf{E}_{s\sim\pi(q(\theta))}[\mathbf{E}[q|s]]$.

Analogous to the standard definition, say a quality function $q\colon \theta \to Q$ is implementable by a direct mechanism $(q(\theta), w(\theta))$ if they satisfy the incentive compatibility constraint

$$w(\theta) - c(q(\theta), \theta) \geq w(\theta') - c(q(\theta'), \theta) \quad \text{for all } \theta, \theta' \in [\underline{\theta}, \bar{\theta}]. \tag{4}$$

The following lemma establishes the equivalence between the direct mechanism and the rating mechanism in terms of implementability and allows one to focus on feasible direct mechanisms $(q(\theta), w(\theta))$.

Lemma 2. *An allocation $q(\theta)$ is implementable by the rating scheme $\pi(q)$ if and only if it is implementable by a feasible direct mechanism $(q(\theta), w(\theta))$.*[34]

Proof. ($\Longrightarrow$) is by the revelation principle and definition of feasibility. ($\Longleftarrow$) is similar to the taxation principle. Construct a $\pi(q)$ that penalizes off-path deviations to q that no types choose in the direct mechanism, so that they will never be chosen in the rating scheme $\pi(q)$ either. □

Remark 1. The lemma implies that eliciting the agent's information through a menu of tests $\Pi \equiv \{\pi\colon Q \to \Delta(S)\}$ has no value. To see this, when a menu of tests is offered, consider a direct mechanism consisting of $q\colon [\underline{\theta}, \bar{\theta}] \to Q$ and $\hat{\pi}\colon [\underline{\theta}, \bar{\theta}] \to \Pi$, along with the induced interim wage $w(\theta) = \mathbf{E}_{s\sim\hat{\pi}(\theta)(q(\theta))}[\mathbf{E}[q|s]]$. The same $w(\theta)$ and $q(\theta)$ can be induced by a rating scheme such that $\pi(q(\theta)) = \hat{\pi}(\theta)(q(\theta))$ for all $\theta \in [\underline{\theta}, \bar{\theta}]$, so it can also be implemented by a single test π.

[34]The implementation result is reminiscent of the Laffont-Tirole cost reimbursement model (Laffont and Tirole, 1993, Chapter 1), where the optimal allocation implemented by a direct mechanism (analogous to $(q(\theta), w(\theta))$) can also be implementable by a payment scheme (analogous to $\hat{w}(q)$), and vice versa.

By the standard argument, incentive compatibility of a direct mechanism $(q(\theta), w(\theta))$ is equivalent to the monotonicity of $q(\theta)$ and the envelope condition

$$w(\theta) - c(q(\theta), \theta) = -\int_{\underline{\theta}}^{\theta} c_\theta(q(x), x)dx + \underline{U} \tag{5}$$

where $\underline{U} = w(\underline{\theta}) - c(q(\underline{\theta}), \underline{\theta})$ (see Lemma B.1). However, an incentive-compatible $(q(\theta), w(\theta))$ is not necessarily feasible because it may be unable to be induced by a rating scheme. The following proposition characterizes the necessary and sufficient condition for the feasibility of an incentive-compatible direct mechanism.

Proposition 1 (Saeedi and Shourideh, 2020, Proposition 1 and Theorem 1[35])**.** *An incentive-compatible direct mechanism $(q(\theta), w(\theta))$ is feasible if and only if $w(\theta)$ is a mean-preserving spread of $v(q(\theta))$ in the quantile space, that is,*

(i) $\int_{\underline{\theta}}^{\theta} w(\theta')\,\mathrm{d}F(\theta') \geq \int_{\underline{\theta}}^{\theta} q(\theta')\,\mathrm{d}F(\theta')$ *for all* $\theta \in [\underline{\theta}, \bar{\theta}]$ *(MPS),*

(ii) $\int_{\underline{\theta}}^{\bar{\theta}} w(\theta)\,\mathrm{d}F(\theta) = \int_{\underline{\theta}}^{\bar{\theta}} q(\theta)\,\mathrm{d}F(\theta)$ *(Bayesian plausibility, henceforth BP).*

The proposition is reminiscent of the symmetric version of Border's theorem (i.e., Maskin-Riley condition) in reduced-form mechanism design (Maskin and Riley, 1984; Border, 1991).[36] Analogously, it allows us to optimize over direct mechanisms $(q(\theta), w(\theta))$ (or interim wages $\hat{w}(q)$) induced by experiments rather than experiments themselves.

Corollary 1.1. *If an incentive-compatible direct mechanism $(q(\theta), w(\theta))$ satisfies $w'(\theta) \leq q'(\theta)$ on $[\underline{\theta}, \bar{\theta}]$, then it is feasible if and only if it satisfies* (BP)*.*

Proof sketch. If $w'(\theta) \leq q'(\theta)$ on $[\underline{\theta}, \bar{\theta}]$, then (BP) implies (MPS) because w single-crosses q from above. □

Because incentive-compatibility implies $w'(\theta) = c_q(q(\theta), \theta)q'(\theta)$, the corollary implies that if an incentive-compatible direct mechanism $(q(\theta), w(\theta))$ satisfies $c_q(q(\theta), \theta) \leq 1$, then it is feasible if and only if it satisfies (BP).

Example 3.1. If $w'(\theta) \leq q'(\theta)$ on $[\underline{\theta}, \bar{\theta}]$, one can construct a rating that induces $(q(\theta), w(\theta))$ à la Albano and Lizzeri (2001). Define q° as a fixed point of $\hat{w}(q)$ (which exists because

[35]See also Rodina (2020, Lemma 1 and Lemma 2) and Saeedi and Shourideh (2023, Lemma 1 and Theorem 1) for a characterization theorem when investing effort increases quality stochastically.

[36]See also Matthews (1984); Kleiner, Moldovanu and Strack (2021). Indeed, it can be proven à la the proof of Border's theorem in Kleiner, Moldovanu and Strack (2021, Theorem 3).

$\hat{w}'(q) \leq 1$) and $p(q) = \frac{q-\hat{w}(q)}{q-q^\circ}$ for $q \neq q^\circ$ and $p(q^\circ) = 0$. Then, $\hat{w}(q)$ can be induced by the rating $\pi\colon Q \to \Delta(S)$ consisting of $S = [q(\underline{\theta}), q(\bar{\theta})] \cup \{\text{pass}, \text{fail}\}$ and

$$\pi(q) = \begin{cases} q & \text{w.p. } 1 - p(q) \\ \text{pass} & \text{w.p. } p(q) \end{cases}, \quad \forall q \in [q(\underline{\theta}), q(\bar{\theta})],$$

$$\pi(q) = \text{fail w.p. } 1, \quad \forall q \notin [q(\underline{\theta}), q(\bar{\theta})].$$

3.2 Principal's Problem

The principal's problem can now be formulated as a mechanism design problem without direct flexible transfers but with a "quasi-transfer" $w(\theta)$ subject to the feasibility constraint because it must be induced by a rating scheme.

$$\text{[P]} \qquad \max_{q(\theta), w(\theta)} \int_{\underline{\theta}}^{\bar{\theta}} v(q(\theta), \theta) \, \mathrm{d}F(\theta) \tag{6}$$

subject to, for all $\theta \in [\underline{\theta}, \bar{\theta}]$,

$$q(\theta) \text{ increasing} \tag{IC-Mon}$$

$$w(\theta) - c(q(\theta), \theta) = -\int_{\underline{\theta}}^{\theta} c_\theta(q(x), x) \, \mathrm{d}x + \underline{U} \tag{IC-Env}$$

$$w(\theta) - c(q(\theta), \theta) \geq 0 \tag{IR}$$

$$\int_{\underline{\theta}}^{\theta} w(\theta') \, \mathrm{d}F(\theta') \geq \int_{\underline{\theta}}^{\theta} q(\theta') \, \mathrm{d}F(\theta'), \tag{MPS}$$

$$\int_{\underline{\theta}}^{\bar{\theta}} w(\theta) \, \mathrm{d}F(\theta) = \int_{\underline{\theta}}^{\bar{\theta}} q(\theta) \, \mathrm{d}F(\theta) \tag{BP}$$

It is worth nothing that the Myersonian approach (i.e., substituting (IC-Env) into the objective, maximizing the resulting expression, and checking the monotonicity of the solution) is infeasible here because there is no transfer function available to ensure that the solution satisfies the incentive compatibility (see also Amador and Bagwell, 2013).

4 Optimal Deterministic Ratings

4.1 Principal's Problem

In this section, I restrict attention to *deterministic* rating schemes $\pi\colon Q \to S$, which either fully reveal the quality or pool some qualities into a single score. By the revelation principle, it is without loss to restrict attention to right-continuous $\pi\colon Q \to S$ because rating schemes that are not right-continuous cannot implement any quality scheme $q(\theta)$ (i.e., no equilibrium exists).[37] When quality is fully revealed, the market learns the quality. When multiple qualities are mapped to the same score s, the lowest quality $\min\{q : \pi(q) = s\}$ (which exists by the right-continuity of π) strictly dominates all other $q \in \{q : \pi(q) = s\}$, so only the lowest quality will be chosen, and the market also learns the quality (see also Zapechelnyuk, 2020, Claim 1). Therefore, in either case, the interim wage is $w(\theta) = q(\theta)$.

Lemma 3. *Under deterministic ratings, the interim wage function is* $w(\theta) = q(\theta)$.

By the revelation principle and Lemma 2, looking for the optimal deterministic rating scheme π is equivalent to looking for the optimal quality scheme $q(\theta)$. Thus, I shall focus on the quality scheme and be fairly casual in distinguishing the two.

Assume $c(q,\theta) = c(q)/\theta$ and rewrite the agent's utility as $u(q,\theta) = \theta q - c(q)$. Substituting $w(\theta) = q(\theta)$, the principal's problem [P] becomes

$$\text{[P']} \qquad \max_{q(\theta)} \int_{\underline{\theta}}^{\bar{\theta}} v(q(\theta),\theta)\,\mathrm{d}F(\theta) \tag{7}$$

subject to, for all $\theta \in [\underline{\theta}, \bar{\theta}]$,

$$q(\theta) \text{ increasing} \qquad \text{(IC-Mon)}$$

$$\theta q(\theta) - c(q(\theta)) = \int_{\underline{\theta}}^{\theta} q(x)\,\mathrm{d}x + \underline{U} \qquad \text{(IC-Env)}$$

$$\theta q(\theta) - c(q(\theta)) \geq 0. \qquad \text{(IR)}$$

The principal's problem [P'] is equivalent to the delegation problem (à la Holmstrom's (1984)) with voluntary participation where the principal determines a set of permissible qualities q and delegates the agent to choose one from the set or the outside option

[37]For example, $\pi(q) = \begin{cases} 0, & \text{if } q \leq 1 \\ 1, & \text{if } q > 1 \end{cases}$ cannot implement any quality scheme $q(\theta)$ because the agent will always choose the quality $q > 1$ that is as close to 1 as possible.

$q = 0$ (or not taking the test) (see Amador and Bagwell, 2022). Indeed, by fully revealing quality, the principal imposes no restrictions on the delegation set. By pooling multiple qualities to the same score, the principal effectively removes all but the lowest of the these qualities from the delegation set. Hence, because choosing a quality q is equivalent to reporting a type θ, the delegation set is $q(\Theta) \equiv \{q(\theta) : \theta \in \Theta\}$.

The following lemma characterizes the properties of the optimal quality scheme $q(\theta)$, which consists of pooling and full revealing intervals and contains at most countably many jump discontinuities.

Lemma 4 (Melumad and Shibano, 1991, Proposition 1; Alonso and Matouschek, 2008, Lemma 2)**.** *An incentive-compatible quality scheme $q(\theta)$ consists of pooling intervals (where $q(\theta)$ is constant) and full revealing intervals (where $q(\theta) = q_f(\theta)$) and at most countably many jump discontinuities.*

At each discontinuity $\hat{\theta} \in [\underline{\theta}, \bar{\theta}]$, the following conditions must hold.

1. $q(\hat{\theta}-) - c(q(\hat{\theta}-), \hat{\theta}) = q(\hat{\theta}+) - c(q(\hat{\theta}+), \hat{\theta})$,
2. $q(\theta) = q(\hat{\theta}-)$ *for* $\theta \in [q_f^{-1}(q(\hat{\theta}-)), \hat{\theta})$ *and* $q(\theta) = q(\hat{\theta}+)$ *for* $\theta \in (\hat{\theta}, q_f^{-1}(q(\hat{\theta}+))]$,
3. $q(\hat{\theta}) \in \{q(\hat{\theta}-), q(\hat{\theta}+)\}$,

where $q_f^{-1}(\cdot) = \max\{\min\{c'(\cdot), \bar{\theta}\}, \underline{\theta}\}$, *and* $q(\hat{\theta}-)$ *and* $q(\hat{\theta}+)$ *denote the left and right limit of* $q(\theta)$ *at* $\hat{\theta}$.

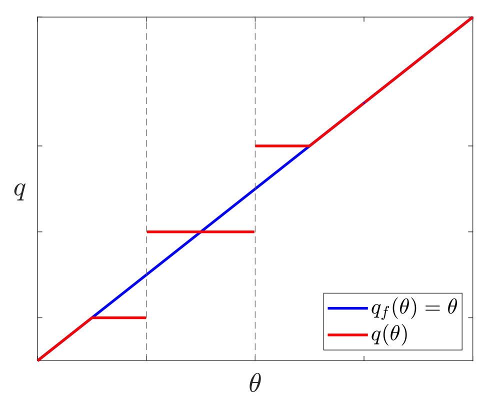


Figure 1: Example of an implementable $q(\theta)$ when $c(q) = q^2/2$

Figure 1 illustrates a quality scheme $q(\theta)$ implementable by a deterministic rating scheme. For example, when $\underline{\theta} = 0$ and $c(q) = q^2/2$, and if $q(\bar{\theta}) = q_f(\bar{\theta})$, an undominated

$q(\theta)$ must be an extreme point of the mean-preserving contraction (MPC) of $q_f(\theta) = \theta$ on $[0, \bar{\theta}]$.[38]

Among the possible jump discontinuities of $q(\theta)$, the jump at the cutoff type due to the participation constraint is of particular interest.

Lemma 5. *There exists a cutoff type $\theta_0 \in [\underline{\theta}, \bar{\theta}]$ such that $q(\theta) = 0$ for all $\theta \in [\underline{\theta}, \theta_0)$ and $q(\theta) > 0$ for all $\theta \in (\theta_0, \bar{\theta}]$. If $\theta_0 \in (\underline{\theta}, \bar{\theta})$, then θ_0 is indifferent between $q_i(\theta_0)$ and $q = 0$; therefore, $q^*(\theta_0) = q_i(\theta_0)$.*

When $\theta_0 \geq \underline{\theta}$, types $\theta \in [\underline{\theta}, \theta_0]$ chooses $q = 0$ and are thus excluded. When $\theta_0 < \underline{\theta}$, the exclusion region $[\underline{\theta}, \theta_0]$ is empty, and all types $\theta \in [\underline{\theta}, \bar{\theta}]$ choose $q > 0$.

Despite the equivalence to delegation, I use a different approach to characterize the solution under weaker sufficient conditions, which are also necessary under linear delegation. The conditions will allow for the optimality of a bang-bang quality scheme (induced by pass/fail tests), where agents either choose $q = 0$ or are bunched at another quality level. See Appendix A.1 (see also Xiao (2023a)).

4.2 When are lower censorship and pass/fail optimal?

Now I focus on two classes of deterministic ratings with minimum standard—lower censorship and pass/fail tests—and characterize conditions for their optimality.

Definition 3. *Lower censorship* is a deterministic rating $\pi\colon Q \to S = Q \cup \{\text{fail}\}$ that reveals the quality q if $q \geq q_0$ for some $q_0 \in Q$ and gives a "fail" otherwise. In other words,

$$\pi(q) = \begin{cases} q, & \text{if } q \geq q_0, \\ \text{fail}, & \text{otherwise.} \end{cases}$$

Definition 4. A *pass/fail test* is a deterministic rating $\pi\colon Q \to \{\text{pass}, \text{fail}\}$ that gives a "pass" if $q \geq q_0$ for some $q_0 \in Q$ and a "fail" otherwise. In other words, $\pi(q) =$

$$\begin{cases} \text{pass}, & \text{if } q \geq q_0, \\ \text{fail}, & \text{otherwise.} \end{cases}$$

The threshold q_0 in these definitions is called a *minimum standard.* A *fully revealing test* is a special case of lower censorship where the minimum standard $q_0 = 0$.

[38]Formally, according to Kleiner et al. (2021, Proposition 5), an undominated $q(\theta)$ is implementable by deterministic ratings if and only if there exists an extension $\tilde{q}(\theta)$ of $q(\theta)$ to $[0, \bar{\theta}']$ such that $\tilde{q}(\bar{\theta}') = q_f(\bar{\theta}')$ ($q(0) = q_f(0)$ is already implied by (IR)) and $\tilde{q}(\theta)$ is an extreme point of the MPC of $q_f(\theta)$ on $[0, \bar{\theta}']$. In general, by the convexity of $c(q)$, Lemma 4 implies that an undominated $q(\theta)$ must be a contraction of $q_f(\theta)$ (but not necessarily mean-preserving) on $[0, \bar{\theta}']$.

Lower censorship with minimum standard $q_0 \in Q$ induces a quality scheme that potentially consists of exclusion, bunching, and fully revealing regions, which takes the form of[39]

$$q(\theta) = \begin{cases} 0, & \text{if } \theta \in [\underline{\theta}, \theta_0) \\ q_i(\theta_0), & \text{if } \theta \in [\theta_0, \theta_c(\theta_0)) \\ q_f(\theta), & \text{if } \theta \in [\theta_c(\theta_0), \bar{\theta}] \end{cases} \tag{8}$$

where $\theta_0 = c(q_0)/q_0 \in \mathbb{R}_+$ that satisfies $q_i(\theta_0) = q_0$ is the cutoff type, and $\theta_c(\theta_0) \equiv q_f^{-1}(q_i(\theta_0)) \in \mathbb{R}_+$ is the type that would choose $q_i(\theta_0)$ under full revelation.

In words, the quality scheme $q(\theta)$ induced by lower censorship with minimum standard q_0 consists of possibly three regions: (i) the *exclusion* region $[\underline{\theta}, \theta_0)$ where agents choose $q = 0$ (and do not take the test), (ii) the *bunching* region $[\theta_0, \theta_c(\theta_0))$ where agents are bunched at $q_0 = q_i(\theta_0)$, and (iii) the *fully revealing* region $[\theta_c(\theta_0), \bar{\theta}]$ where agents choose $q_f(\theta)$. Some of these regions can be empty if (i) $\theta_0^* \leq \underline{\theta}$, (ii) $\theta_c(\theta_0^*) \leq \underline{\theta}$, or (iii) $\theta_c(\theta_0^*) \geq \bar{\theta}$. It is useful to define the start of fully revealing region by $\theta_L(\theta_0) = \text{med}\{\theta_c(\theta_0), \underline{\theta}, \bar{\theta}\}$.[40]

Analogously, a pass/fail test with minimum standard $q_0 \in Q$ induces the quality scheme in the form of

$$q(\theta) = \begin{cases} 0, & \text{if } \theta \in [\underline{\theta}, \theta_0) \\ q_i(\theta_0), & \text{if } \theta \in [\theta_0, \bar{\theta}] \end{cases}$$

where $\theta_0 = c(q_0)/q_0 \in \mathbb{R}_+$ that satisfies $q_i(\theta_0) = q_0$ is the cutoff type. A pass/fail induces a quality scheme $q(\theta)$ that consists of possibly two regions: (i) the *exclusion* region $[\underline{\theta}, \theta_0)$ where agents choose $q = 0$ and (ii) the *bunching* region $[\theta_0, \bar{\theta}]$ where agents choose $q_0 = q_i(\theta_0)$.

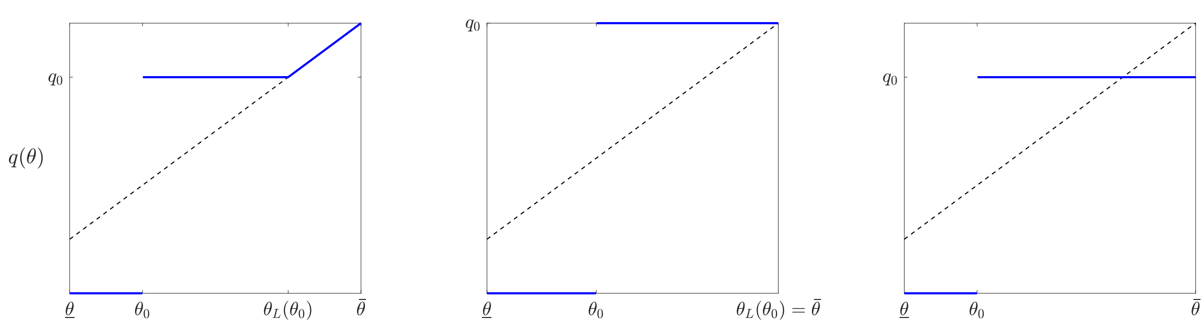


Figure 2: $q(\theta)$ induced by lower censorship (left, center) and pass/fail (center, right)

There are several caveats. First, it is possible that $c(q_0)/q_0 \in [0, \underline{\theta}]$. In this case, $\theta_0 =$

[39]By convention, $[x, y)$, (x, y), and $[x, y]$ represent the empty set if $x \geq y$.

[40]Define $\text{med}\{x, y, z\}$ as the median of x, y, and z. In other words, $\theta_L(\theta_0) = \max\{\min\{\theta_c(\theta_0), \bar{\theta}\}, \underline{\theta}\}$.

$c(q_0)/q_0$ is a hypothetical cutoff "type" that is below $\underline{\theta}$, and the exclusion region $[\underline{\theta}, \theta_0)$ is empty.

Second, it is also possible that $c(q_0)/q_0 \geq \bar{\theta}$. However, because any $q_0 > q_i(\bar{\theta})$ cannot be optimal because no one would meet this standard, it is without loss to assume $q_0 \leq q_i(\bar{\theta})$ (and I have already defined $q_{\max} = q_i(\bar{\theta})$). Thus, for any $q_0 \in Q$, there exists $\theta_0 = c(q_0)/q_0 \in [0, \bar{\theta}]$ such that $q_i(\theta_0) = q_0$.[41]

Third, for lower censorship with minimum standard $q_0 \geq q_f(\bar{\theta})$, we have $\theta_c(\theta_0) \geq \bar{\theta}$, so the fully revealing region $[\theta_c(\theta_0), \bar{\theta}]$ is empty. In words, the minimum standard is so high that no one will choose any quality strictly above it in equilibrium. Thus, the lower censorship induces the same quality scheme as a pass/fail test with the same minimum standard q_0.[42] On the other hand, if $\theta_c(\theta_0) < \bar{\theta}$, $q(\theta)$ is continuous at $\theta_c(\theta_0)$ because $q_i(\theta_0) = q_f(\theta_c(\theta_0))$ by definition.

Lastly, the exclusion region is empty if and only if $\theta_0 \leq \underline{\theta}$, and both exclusion and bunching regions are empty if and only if $\theta_c(\theta_0) \leq \underline{\theta}$.

4.2.1 Quality Maximization

To provide intuitions, I start with the simplest case where the principal's objective is expected quality, i.e., $v(q, \theta) = q$. In this case, the sufficient conditions for the optimality of lower censorship and pass/fail tests depend only on the probability density $f(\theta)$.

Proposition 2. *Assume $v(q, \theta) = q$. The optimal deterministic rating scheme*

- *is lower censorship if $f(\theta)$ is unimodal,*
- *is pass/fail if $f(\theta)$ is increasing or $\theta_c(\underline{\theta}) \geq \bar{\theta}$,*
- *induces no exclusion if $f(\theta)$ is decreasing,*
- *is fully revealing if and only if $f(\theta)$ is decreasing and $\underline{\theta} = 0$.*

Denote by $\theta_m \in [\underline{\theta}, \bar{\theta}]$ the mode of $f(\theta)$. Then, the optimal cutoff type $\theta_0^* \in [\theta_c^{-1}(\theta_m), \theta_m]$.

Proof. The proof follows immediately from the necessary and sufficient conditions in Proposition 3 for the more general case. □

The optimal deterministic rating has a minimum standard $q_0 = q_i(\theta_0^*)$ above which it fully reveals quality. The rating scheme leads to (i) exclusion of the low types $[\underline{\theta}, \theta_0^*)$, (ii)

[41] For $q_0 = 0$, $\theta_0 = \lim_{q_0 \to 0} c(q_0)/q_0 = 0$.

[42] Note that although they induce the same quality scheme in equilibrium, a pass/fail test is not a special case of lower censorship because the off-path strategies $q > q_0$ lead to different outcomes.

pooling of the intermediate types $[\theta_0^*, \theta_L(\theta_0^*))$ who are bunched at the minimum standard $q_i(\theta_0^*)$, and (iii) full separation of the high types $[\theta_L(\theta_0^*), \bar{\theta}]$.[43] In particular, if $\theta_c(\theta_0^*) \geq \bar{\theta}$, the optimal deterministic rating is a pass/fail test. See Figure 3 for an illustration of the optimal quality scheme $q^*(\theta)$ for decreasing, unimodal, and increasing $f(\theta)$. Importantly, the mode θ_m of the density $f(\theta)$ must be in the bunching region, i.e., $\theta_m \in [\theta_0^*, \theta_L(\theta_0^*)]$.

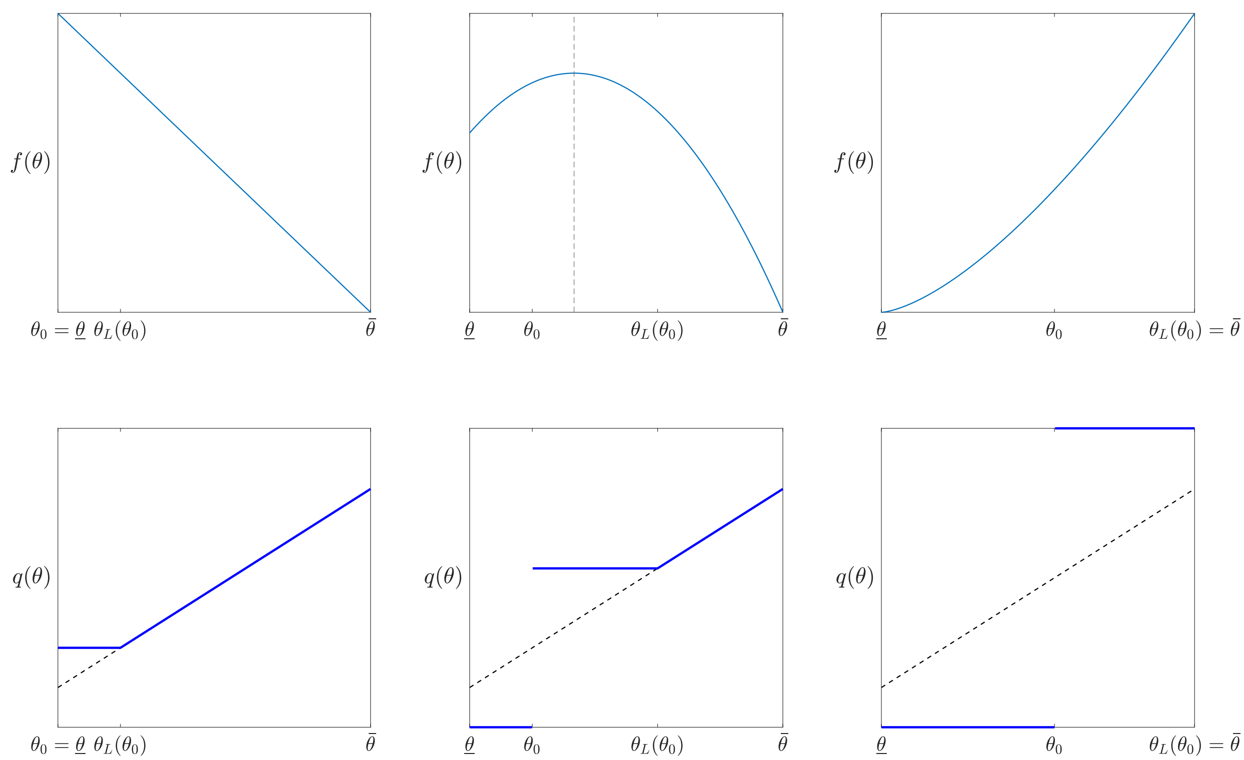


Figure 3: $q^*(\theta)$ for decreasing, unimodal, and increasing $f(\theta)$

Intuition. First, consider a perturbation to $q^*(\theta) = q_f(\theta)$ in the fully revealing region at $\hat{\theta} \in (\theta_L(\theta_0^*), \bar{\theta})$. By Lemma 4, the perturbation leads to

$$q_\varepsilon(\theta) = \begin{cases} q_f(\hat{\theta} - \varepsilon), & \text{if } \theta \in (\hat{\theta} - \varepsilon, \hat{\theta}), \\ q_f(\hat{\theta} + \varepsilon), & \text{if } \theta \in (\hat{\theta}, \hat{\theta} + \varepsilon). \end{cases}$$

By setting a minimum standard $q_f(\hat{\theta} + \varepsilon)$, the rating scheme creates two pooling regions: $[\hat{\theta} - \varepsilon, \hat{\theta}]$ (lower types) and $[\hat{\theta}, \hat{\theta} + \varepsilon]$ (higher types). Thus, the minimum standard leads

[43] Some of these intervals can be empty if (i) $\theta_0^* \leq \underline{\theta}$, (ii) $\theta_c(\theta_0^*) \leq \underline{\theta}$, or (iii) $\theta_c(\theta_0^*) \geq \bar{\theta}$.

to a trade-off: on the one hand, it induces higher types to invest more in quality than they would under full revelation to separate themselves from the lower types who cannot meet the standard. On the other hand, it discourages the lower types from investing in quality because they will not reach the minimum standard (as participation is voluntary).

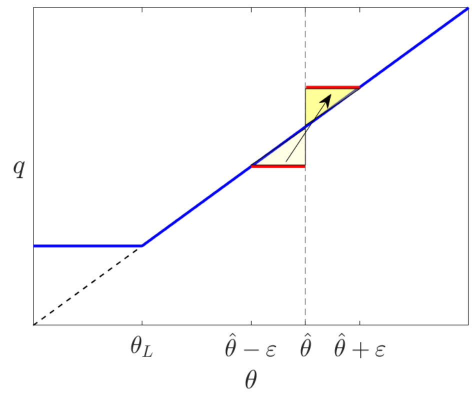


Figure 4: A perturbation (in red) to $q_f(\theta)$ at $\hat{\theta}$ in the fully revealing region

Figure 4 illustrates this trade-off. Specifically, when $v(q,\theta) = q$ and $c(q) = q^2/2$, the loss due to discouraged lower types is represented by the area of the triangle on the left (in light yellow), and the gain due to the motivated higher types is represented by the area of the triangle on the right (in bright yellow). The two triangles have the same area.[44] Therefore, the shift of the area from the left to the right decreases average quality (i.e., the loss from lower types exceeds the gain from higher types) if and only if the density $f(\theta)$ is decreasing at $\hat{\theta}$ so that the area on the left has more weight. In other words, the principal will fully reveal quality (and not set a minimum standard) in the region where $f(\theta)$ is decreasing.

On the other hand, if the density $f(\theta)$ is increasing on $[\underline{\theta}, \bar{\theta}]$, the gain from higher types always exceeds the loss from lower types, even as perturbations become large, because the two triangles always have the same area. Thus, the optimal rating does not induce any fully revealing region, and agents either choose $q = 0$ or the minimum standard. In other words, if $f(\theta)$ is increasing, the principal will set a high minimum standard such that even the highest type needs to invest more in quality than he would under full revelation to pass the test, in order to provide stronger incentives to high types at the cost of excluding more low types. Because types are increasingly more concentrated towards the top, this simple pass/fail test induces the highest average quality.

[44] For general $c(q)$, the loss and the gain regions still have the same area, although they are not necessarily triangles.

Second, consider a perturbation to the optimal scheme $q^*(\theta)$ in the exclusion and bunching regions. The perturbation can involve either a lower or higher minimum standard, which leads to more or less participation. Intuitively, a lower minimum standard increases participation because more lower types can reach the standard without violating their participation constraints. On the other hand, it reduces the incentives for higher types who are bunched at the minimum standard. Analogously, a higher minimum standard reduces participation but increases the incentives for higher types to invest in quality.

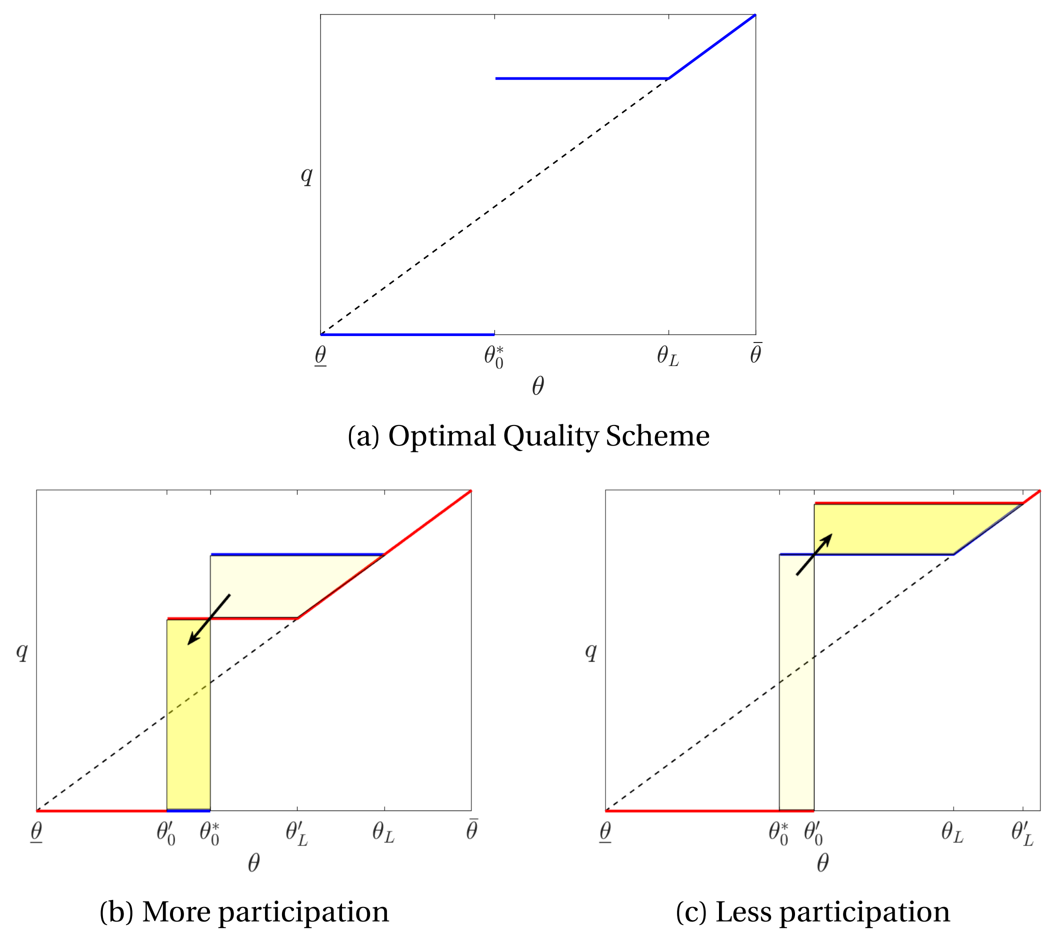


Figure 5: Perturbations on pooling regions

Figure 5 illustrates this trade-off in both directions. Similar to a perturbation in the fully revealing region, the loss (in light yellow) and the gain (in bright yellow) have the same area. Note that if $f(\theta)$ is unimodal with a mode $\theta_m \in [\underline{\theta}, \bar{\theta}]$, the optimal cutoff θ_0^* is such that $\theta_m \in [\theta_0^*, \theta_L(\theta_0^*)]$ is in the bunching region. Thus, unimodality of the density implies either more or less participation is undesirable. In particular, if $f(\theta)$ is decreasing on $[\underline{\theta}, \bar{\theta}]$, no exclusion is optimal (i.e., $\theta_0^* = \underline{\theta}$) because reducing participation for a higher minimum standard (Figure 5b) is undesirable for the same argument as before. If $f(\theta)$ is increasing, $\theta_L(\theta_0^*) \geq \theta_m = \bar{\theta}$ implies $\theta_L(\theta_0^*) = \bar{\theta}$, so a pass/fail test is optimal, as argued before. Alternatively, if $\theta_c(\underline{\theta}) \geq \bar{\theta}$, then $\theta_c(\theta_0^*) \geq \bar{\theta}$ for every possible $\theta_0^* \in [\underline{\theta}, \bar{\theta}]$. In other words, the range of types is so small that the start of the fully revealing region is higher than $\bar{\theta}$, so the fully revealing region can never be reached. Hence, pass/fail test is also optimal in this case.[45]

4.2.2 Linear Delegation

In this subsection, I extend the analysis to a more general class of objective functions, $v(q, \theta) = \beta(\theta)q - \alpha c(q) + d(\theta)$. This is referred to as "linear delegation" in Kolotilin and Zapechelnyuk (2019) because the principal's marginal payoff from the agent's action q is linear in (a transformation of) the agent's action. In this case, the "relative concavity" of the principal and agent's preferences, given by $-v_{qq}(q, \theta)/c''(q) = \alpha$, is constant.

Condition (LD). The principal's objective function is $v(q, \theta) = \beta(\theta)q - \alpha c(q) + d(\theta)$, where $\alpha \geq 0$ and $\beta(\theta) \geq \alpha\theta$ (by Assumption 1).

Necessary and Sufficient Conditions

Define the characteristic function $r(\theta)$ and $R(\theta)$, which generalizes the density $f(\theta)$ and distribution $F(\theta)$ by incorporating the principal and agent's preferences, as

$$r(\theta) = (\beta(\theta) - \alpha\theta)f(\theta) - \alpha(F(\theta) - F(\theta_0)) \text{ on } \mathbb{R}_+ \tag{9}$$

and

$$R(\theta) = \int_{\underline{\theta}}^{\theta} r(\tilde{\theta})\, \mathrm{d}\tilde{\theta} = \int_{\underline{\theta}}^{\theta} \beta(\tilde{\theta})f(\tilde{\theta})\, \mathrm{d}\tilde{\theta} - \alpha\theta(F(\theta) - F(\theta_0)). \tag{10}$$

Note that $r(\theta)$ is defined on $\mathbb{R}_+$ instead of $[\underline{\theta}, \bar{\theta}]$, which requires extending $F(\theta)$ and $f(\theta)$ from $[\underline{\theta}, \bar{\theta}]$ to $\mathbb{R}_+$. By convention, for all $\theta < \underline{\theta}$, $f(\theta) = 0$, $F(\theta) = 0$, and thus $r(\theta) = \alpha F(\theta_0) \geq 0$; for all $\theta > \bar{\theta}$, $f(\theta) = 0$, $F(\theta) = 1$, and thus $r(\theta) = -\alpha(1 - F(\theta_0)) \leq 0$.

Example (Quality Maximization)**.** If $v(q, \theta) = q$, then $r(\theta) = f(\theta)$ and $R(\theta) = F(\theta)$.

Observation 1. (i) $r(\theta) \geq 0$ for all $\theta \leq \theta_0$, so R is increasing on $[0, \theta_0]$. (ii) $r(\theta)$ can be discontinuous at $\underline{\theta}$ (if $\underline{\theta} > 0$) and $\bar{\theta}$ because $r(\underline{\theta}) \geq r(\underline{\theta}-) = \alpha F(\theta_0)$ and $r(\bar{\theta}) \geq r(\bar{\theta}+) = -\alpha(1 - F(\theta_0))$. In these cases, R can be non-differentiable and have a convex kink at $\underline{\theta}$ and a concave kink at $\bar{\theta}$.[46]

[45] Cf. Zapechelnyuk (2020, Theorem 2). See Remark 4 for the comparison.
[46] See also Lemma B.3 for discontinuities for general preferences.

The characterization function $R(\theta)$ is determined by the distribution $F(\theta)$, objective $v(q,\theta)$, the cost function $c(q)$. It can be viewed as a quasi-distribution function of types.

First, I focus on the exclusion and bunching regions $[\underline{\theta}, \theta_c(\theta_0)]$. Define the multiplier $A(\theta_0)$ by

$$A(\theta_0) = \frac{1}{\theta_c(\theta_0) - \theta_0} \int_{\theta_0}^{\theta_c(\theta_0)} r(\theta)\, \mathrm{d}\theta, \tag{11}$$

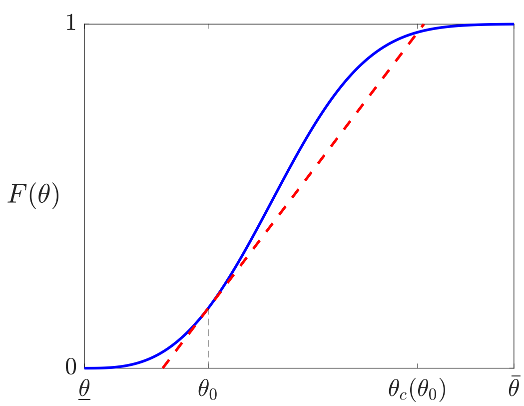
which is the slope of the line connecting θ_0 and $\theta_c(\theta_0)$ on $R(\theta)$. In particular, if $\theta_c(\theta_0) = \theta_0$ (i.e., $\theta_0 = 0$), $A(\theta_0) = \lim_{\theta \to \theta_0^+} \frac{R(\theta) - R(\theta_0)}{\theta - \theta_0} = r(\theta_0+)$.

Example (Quadratic cost)**.** If $c(q) = q^2/2$, then $\theta_c(\theta_0) = 2\theta_0$, and $A(\theta_0) = \frac{R(2\theta_0) - R(\theta_0)}{\theta_0}$.

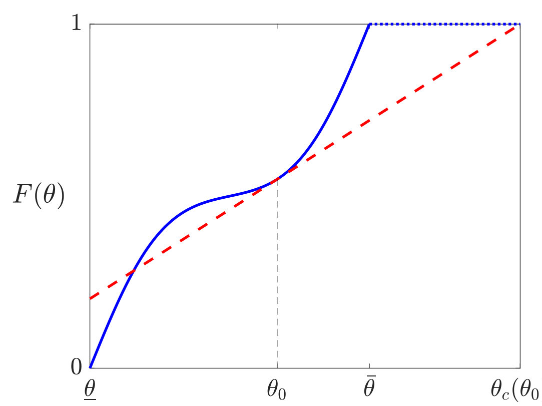
I state a condition on $r(\theta)$ that must hold in the bunching and the exclusion regions.

Condition (S) (Subgradient). $\int_{\theta_0}^{\theta} r(\tilde{\theta})\, \mathrm{d}\tilde{\theta} \geq A(\theta_0)(\theta - \theta_0)$ and $A(\theta_0) > 0$ for all $\theta \in [0, \theta_L(\theta_0)]$.

By the definition of $A(\theta_0)$, condition (S) holds with equality at $\theta = \theta_c(\theta_0)$. Condition (S) says that $A(\theta_0)$ is the *subgradient* of the restriction of the function R to $[0, \theta_c(\theta_0)]$, denoted by $R|_{[0,\theta_c(\theta_0)]}$, at θ_0. Geometrically, this means the line ℓ connecting θ_0 and $\theta_c(\theta_0)$ (red dashed line in Figure 6) lies below R for all $\theta \in [0, \theta_c(\theta_0)]$. In technical terms, ℓ is the supporting hyperplane of the epigraph of $R|_{[0,\theta_c(\theta_0)]}$ containing θ_0. If $R(\theta)$ is differentiable at θ_0, then ℓ must be tangent to $R(\theta)$ at θ_0, i.e., $r(\theta_0) = A(\theta_0)$ (see Lemma B.4).



(a) $F(\theta)$ that satisfies condition (S)



(b) $F(\theta)$ that violates condition (S)

Figure 6: Geometric Illustration of Condition (S)

Observation 2. Condition (S) can never be satisfied at $\theta_0 \geq \bar{\theta}$.

Second, I state another condition on the concavity of $R(\theta)$ in the fully revealing region.

Condition (C) (Concavity). $r(\theta)$ is decreasing in θ on $(\theta_L(\theta_0), \bar{\theta}]$.

To characterize the set of functions that satisfy conditions (S) and (C), I introduce the following definitions that generalize the definitions of unimodal, increasing, and decreasing functions $r(\theta)$.

Definition 5. A function $r(\theta)$ is

- *quasi-unimodal* if it satisfies conditions (S) and (C) for some θ_0,
- *quasi-increasing* if it satisfies condition (S) at some θ_0 such that $\theta_c(\theta_0) \geq \bar{\theta}$ (i.e., $q_i(\theta_0) > q_f(\bar{\theta})$),
- *quasi-decreasing* if it satisfies conditions (S) and (C) at some $\theta_0 \leq \underline{\theta}$.

Remark 2. Quasi-I on $\mathbb{R}_+$ is equivalent to quasi-I on $[\underline{\theta}, \bar{\theta}]$, for $I \in \{$unimodal, increasing, decreasing$\}$, which make the statements convenient. Nevertheless, I on $\mathbb{R}_+$ is stronger than I on $[\underline{\theta}, \bar{\theta}]$ because $f(\theta) = 0$ for all $\theta \notin [\underline{\theta}, \bar{\theta}]$.[47]

Example (Linear-Quadratic)**.** Assume $v(q, \theta) = q$, $c(q) = q^2/2$, and $\underline{\theta} = 1$. Figure 7 illustrates quasi-unimodal, quasi-increasing, and quasi-decreasing $f(\theta)$ on $[\underline{\theta}, \bar{\theta}]$.

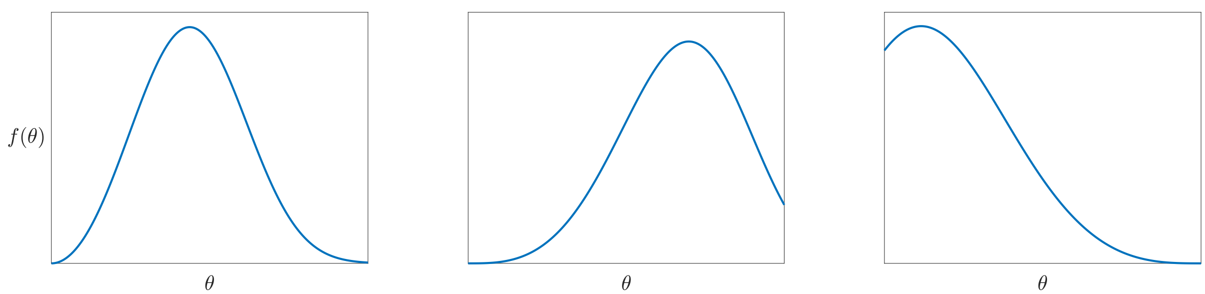


Figure 7: Quasi-unimodal (all), quasi-increasing (center), and quasi-decreasing (right)

In particular, for a left-skewed unimodal $f(\theta)$ (center of Figure 7), Figure 8 illustrates that condition (S) is satisfied at some θ_0^* such that $\theta_c(\theta_0^*) \geq \bar{\theta}$ (so that f is quasi-increasing).

Loosely speaking, $r(\theta)$ is quasi-unimodal if types are concentrated around the mode of $r(\theta)$, quasi-increasing if types are concentrated towards the top of $r(\theta)$ (i.e., r is sufficiently left-skewed), and quasi-decreasing if types are concentrated towards the bottom of $r(\theta)$ (i.e., r is sufficiently right-skewed).[48] In each case, it need not be exactly unimodal, increasing, or decreasing in $r(\theta)$ because the definition allows some wiggle room for

[47] For example, if $\theta \sim$ Unif $[1, 2]$, then $f(\theta) = \mathbf{1}[1 \leq \theta \leq 2]$ is decreasing and increasing on $[1, 2]$ but neither decreasing nor increasing on $\mathbb{R}_+$.

[48] $r(\theta)$ need not be a real probability density function, although $r(\theta) = f(\theta)$ when $v(q, \theta) = q$.

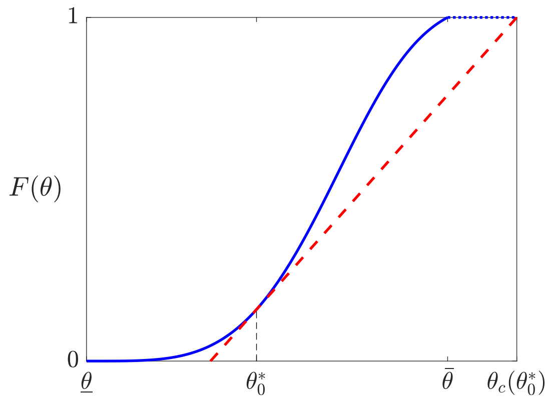


Figure 8: Unimodal f that is quasi-increasing (and quasi-unimodal)

small deviations. The magnitude of deviations allowed depends on $[\underline{\theta}, \bar{\theta}]$ and $\theta_c(\theta)$. For example, if $\underline{\theta} = 0$, then $r(\theta)$ is quasi-decreasing if and only if it is decreasing—i.e., no deviations are allowed. The following lemma formalizes these intuitions.

Lemma 6. *If r is unimodal on $[\underline{\theta}, \bar{\theta}]$, then it is quasi-unimodal. If r is increasing on $[\underline{\theta}, \bar{\theta}]$, then it is quasi-increasing. If r is decreasing on $[\underline{\theta}, \bar{\theta}]$, then it is quasi-decreasing; the converse is true if $\underline{\theta} = 0$. If $\bar{\theta} \leq \theta_c(\underline{\theta})$, then every unimodal $r(\theta)$ is quasi-increasing.*

With the definitions above, I can state the sufficient conditions for lower censorship and pass/fail tests conveniently.

Proposition 3 (Necessary and Sufficient Conditions)**.** *Under Condition LD, the optimal deterministic rating scheme*

- *is lower censorship (with cutoff type θ_0^*) if and only if $r(\theta)$ is quasi-unimodal (with conditions* (S) *and* (C) *satisfied at θ_0^*),*
- *is pass/fail if and only if $r(\theta)$ is quasi-increasing,*
- *induces no exclusion if and only if $r(\theta)$ is quasi-decreasing,*
- *is fully revealing if and only if $r(\theta)$ is decreasing on $\mathbb{R}_+$.*

Proof sketch. Sufficiency can be shown using optimal control methods in Appendix A.1. Necessity can be shown à la Amador and Bagwell (2013, Proposition 2) using perturbation methods. □

Remark 3. If $\underline{\theta} = 0$, then lower censorship induces no exclusion if and only if it is fully revealing.

See the proof in Appendix B.2.2. If conditions (S) and (C) are satisfied at θ_0^*, the optimal deterministic rating has a minimum standard $q_0 = q_i(\theta_0^*)$ above which it fully reveals quality. The induced quality scheme $q^*(\theta)$ features (i) exclusion of the low types $[\underline{\theta}, \theta_0^*)$, (ii) pooling of the intermediate types $[\theta_0^*, \theta_L(\theta_0^*))$ who are bunched at the minimum standard $q_i(\theta_0^*)$, and (iii) full separation of the high types $[\theta_L(\theta_0^*), \bar{\theta}]$.[49] In particular, if $\theta_c(\theta_0^*) \geq \bar{\theta}$ (or equivalently, $q_i(\theta_0^*) \geq q_f(\bar{\theta})$), the optimal deterministic rating is a pass/fail test.

The following corollary, which follows immediately from Lemma 6, provides sufficient conditions that are easy to check, as they depend only on the shape of $r(\theta)$ and guarantee the existence of θ_0 that satisfy conditions (S) and (C) without solving for it.

Corollary 3.1 (Sufficient conditions)**.** *The sufficient conditions for lower censorship, pass/fail tests, and lower censorship without exclusion are that $r(\theta)$ is unimodal, increasing, and decreasing, respectively.*

Intuition. Under Condition LD, the "density" function $r(\theta)$ incorporates $\beta(\theta)$ and α into the density function $f(\theta)$. First, when $\alpha = 0$, $\beta(\theta)$ can be easily incorporated into the density, as $\beta(\theta)f(\theta)$ can be treated as the density.[50] Thus, the intuitions for the quality maximization case (in Section 4.2.1) that relates the density to the optimal deterministic rating scheme carry over.

Second, fix $\beta(\theta) = \theta$. Then, when $\alpha = 0$, $\tilde{f}(\theta) \equiv \theta f(\theta)$ can be treated as the density. As α increases from 0 to 1, the principal's preference becomes more aligned with the agent's preference, so a perturbation to the fully revealing region is less likely to be desirable. So is a perturbation on the pooling regions that reduces participation. Thus, the "density" function $r(\theta) = (1-\alpha)\tilde{f}(\theta) - \alpha(F(\theta) - F(\theta_0))$ is more likely to be decreasing as α increases.

Treating $r(\theta)$ as the density, conditions (C) is equivalent to the density being decreasing in the fully revealing region, as discussed in Section 4.2.1. Condition (S), which is weaker than unimodality, is necessary and sufficient for the cutoff to be θ_0^* (see Section 4.2.2). Moreover, the necessary and sufficient condition for pass/fail tests is that $r(\theta)$ is quasi-increasing because the range of type can be so small that the start of a fully revealing region cannot be reached even at $\bar{\theta}$ (i.e., $\theta_c(\theta_0^*) \geq \bar{\theta}$).

Comparison with Amador and Bagwell (2022). I briefly compare my results with Amador and Bagwell (2022, henceforth AB). See Appendix C for a detailed comparison.

[49] Some of these intervals can be empty if (i) $\theta_0^* \leq \underline{\theta}$, (ii) $\theta_c(\theta_0^*) \leq \underline{\theta}$, or (iii) $\theta_c(\theta_0^*) \geq \bar{\theta}$.

[50] This is because, for $v(q,\theta) = \beta(\theta)q$, $\int_{\underline{\theta}}^{\bar{\theta}} v(q,\theta)f(\theta)\,\mathrm{d}\theta = \int_{\underline{\theta}}^{\bar{\theta}} q(\theta)(\beta(\theta)f(\theta))\,\mathrm{d}\theta$.

Define

$$L(\theta|\theta_0) = \frac{R(\theta) - R(\theta_0)}{\theta - \theta_0} = \frac{1}{\theta - \theta_0} \int_{\theta_0}^{\theta} r(\tilde{\theta}) \, \mathrm{d}\tilde{\theta}, \tag{12}$$

which is the slope of the line connecting θ_0 and θ on $R(\theta)$. In particular, $A(\theta_0) = L(\theta_c(\theta_0)|\theta_0)$. Then, condition (S) can be decomposed into two conditions on the bunching and the exclusion regions, respectively.

Condition (S1). $L(\theta|\theta_0) \geq L(\theta_c(\theta_0)|\theta_0) = A(\theta_0) > 0$ for all $\theta \in (\theta_0, \theta_L(\theta_0)]$.

Condition (S2). $L(\theta|\theta_0) \leq L(\theta_c(\theta_0)|\theta_0) = A(\theta_0)$ for all $\theta \in [0, \theta_0)$.

The following two observations summarize the differences between my conditions and AB's sufficient conditions.

Observation 3. Because $\theta_c(\theta_0) \geq \theta_L(\theta_0)$, condition (S1) is weaker than condition (i) in AB, which is equivalent to $L(\theta|\theta_0) \geq L(\theta_L(\theta_0)|\theta_0)$ for all $\theta \in (\theta_0, \theta_L(\theta_0)]$ (see condition AB(i) in Appendix C). Consequently, their condition implies that a fully revealing region must exist because it rules out the possibility that $\theta_c(\theta_0) > \bar{\theta}$ (e.g., when $r(\theta)$ is increasing).

Observation 4. AB also require condition (C) to hold at all $\theta_0 \in [\underline{\theta}, \bar{\theta})$. Then, condition (S) can only hold at $\theta_0 \leq \underline{\theta}$, so no exclusion is always optimal. Therefore, a pass/fail test can never be optimal, except in the trivial case (i.e., $\bar{\theta} \leq \theta_c(\underline{\theta})$) where no type fails in the equilibrium.

Optimal Cutoff Type

As mentioned above, condition (S) ensures that θ_0 is the optimal cutoff type. To see this, under lower censorship (or pass/fail tests if $\theta_L(\theta_0) = \bar{\theta}$) with cutoff type $\theta_0 \in [\underline{\theta}, \bar{\theta}]$, the principal's expected payoff is given by

$$V(\theta_0) = \int_{\theta_0}^{\theta_L(\theta_0)} v(q_i(\theta_0), \theta) \, \mathrm{d}F(\theta) + \int_{\theta_L(\theta_0)}^{\bar{\theta}} v(q_f(\theta), \theta) \, \mathrm{d}F(\theta). \tag{13}$$

By definition, $c(q_i(\theta_0)) = \theta_0 q_i(\theta_0)$ implies $q_i'(\theta_0) = \frac{q_i(\theta_0)}{\theta_c(\theta_0) - \theta_0} > 0$. Thus, a higher cutoff θ_0 is associated with a higher minimum standard $q_i(\theta_0)$, thereby increasing the principal's expected payoff in the bunching region $[\theta_0, \theta_L(\theta_0)]$. At the same time, it implies more exclusion.

Observation 5. The multiplier $A(\theta_0)$ defined in equation (11) is equal to

$$A(\theta_0) = L(\theta_c(\theta_0)|\theta_0) = \frac{1}{\theta_c(\theta_0) - \theta_0} \int_{\theta_0}^{\theta_L(\theta_0)} v_q(q_i(\theta_0), \theta) \, \mathrm{d}F(\theta). \tag{14}$$

Thus, $V'(\theta_0) = A(\theta_0) q_i(\theta_0) - v(q_i(\theta_0), \theta_0) f(\theta_0)$.

If condition (S) is satisfied at θ_0, then θ_0 satisfy the Kuhn-Tucker first-order condition[51]

$$\begin{cases} V'(\theta_0) = A(\theta_0) q_i(\theta_0) - v(q_i(\theta_0), \theta_0) f(\theta_0) = 0 & \text{if } \theta_0 > 0, \\ V'(\theta_0) \leq 0 & \text{if } \theta_0 = 0. \end{cases} \tag{OPT}$$

In words, increasing the cutoff θ_0 leads to a marginal increase of $A(\theta_0) \cdot q_i(\theta_0)$ in the bunching region (due to a higher minimum standard) and a marginal decrease in the principal's expected payoff of $v(q_i(\theta_0), \theta_0) f(\theta_0)$ in the exclusion region (due to more exclusion).[52] The optimal cutoff is when these two marginal effects balance out each other.

The following observations provide sufficient conditions for the optimality of "no rent at the bottom" (i.e., $\theta_0^* \geq \underline{\theta}$) and of no exclusion (i.e., $\theta_0^* \leq \underline{\theta}$).

Observation 6. If $v_q(q_i(\theta), \theta) \geq 0$ for all $\theta \in [\underline{\theta}, \theta_c(\underline{\theta})]$ (and strictly positive for some θ), then the optimal cutoff $\theta_0^* \geq \underline{\theta}$. Therefore, the lowest type has no information rent (i.e., $\underline{U} = 0$).

Intuitively, if $v_q(q_i(\theta), \theta) \geq 0$ for all $\theta \in [\underline{\theta}, \theta_c(\underline{\theta})]$, the principal can always benefit from a higher minimum standard that push the lowest type $\underline{\theta}$ to the boundary of the (IR) condition (without increasing exclusion).

Observation 7. If $f(\theta)$ is decreasing and $v_{q\theta}(q, \theta) \leq -v_{qq}(q, \theta)/c''(q)$ for all $q \in Q$ and $\theta \in [\underline{\theta}, \bar{\theta}]$, then no exclusion is optimal (i.e., $\theta_0^* \leq \underline{\theta}$).

The proof is in Appendix B.2.3. The observations hold for general preferences $v(q, \theta)$, regardless of Condition LD.

4.2.3 Examples: State-(In)Dependent Preferences

In this subsection, I provide several common examples of linear delegation, which include state-independent and state-dependent preferences.

[51] See also the jump (switching) condition in the optimal control in Bryson and Ho (1975, Chapter 3.7) and Clarke (2013, Chapter 22.5).

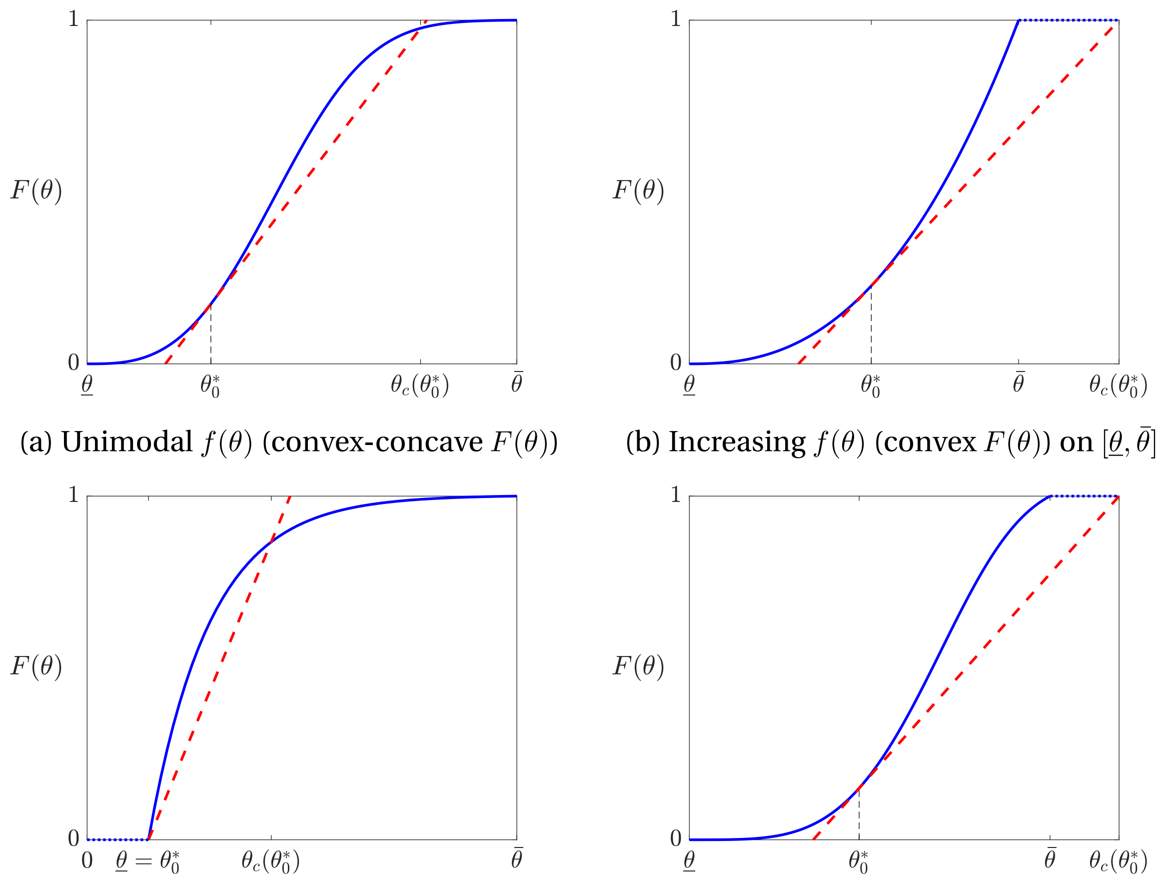
[52] The marginal effects due to a higher $\theta_L(\theta_0)$ always cancel out each other in the bunching and fully revealing regions.

State-Independent Preferences. Quality maximization (i.e., $v(q, \theta) = q$) is commonly studied in the literature (Zapechelnyuk, 2020; Rodina and Farragut, 2020).

Example 4.1 (Quality Maximization)**.** Assume $v(q, \theta) = q$. Thus, we have $r(\theta) = f(\theta)$ and $R(\theta) = F(\theta)$. By Proposition 3, the optimal deterministic rating scheme

- is lower censorship if and only if $f(\theta)$ is quasi-unimodal,
- is pass/fail if and only if $f(\theta)$ is quasi-increasing,
- has no exclusion if and only if $f(\theta)$ is quasi-decreasing, and
- is fully revealing if and only if $\underline{\theta} = 0$ and $f(\theta)$ is decreasing.

The sufficient conditions are $f(\theta)$ being unimodal, increasing, and decreasing on $[\underline{\theta}, \bar{\theta}]$, respectively. See Figure 9 for graphical illustrations (and Figure 7 for the density functions).



(a) Unimodal $f(\theta)$ (convex-concave $F(\theta)$)

(b) Increasing $f(\theta)$ (convex $F(\theta)$) on $[\underline{\theta}, \bar{\theta}]$

(c) Decreasing $f(\theta)$ (concave $F(\theta)$) on $[\underline{\theta}, \bar{\theta}]$

(d) Unimodal $f(\theta)$ that is quasi-increasing

Figure 9: Quality Maximization

Remark 4. Zapechelnyuk (2020, Theorem 2) provides a sufficient condition for pass/fail that is equivalent to $f(\theta)$ being unimodal and $\bar{\theta} \leq \theta_c(\underline{\theta})$, which implies that $f(\theta)$ is quasi-increasing.[53] The latter is restrictive for small $\underline{\theta}$ because $\theta_c(0) = 0$.

In addition to quality maximization, a principal with state-independent preferences may have a constant bliss point $q_e < q_{\max}$. For example, Kartik, Kleiner and Van Weelden (2021) assumes a linear or quadratic loss with a constant bliss point.

Example 4.2 (State-independent Quadratic Loss)**.** Assume $v(q,\theta) = -(q-1)^2$, $u(q,\theta) = -(q-\theta)^2$, and $\Theta = [0,1]$. This is equivalent to $v(q,\theta) = q - q^2/2$ and $u(q,\theta) = \theta q - q^2/2$. Then,

$$r(\theta) = (1-\theta)f(\theta) - (F(\theta) - F(\theta_0))$$

and $R(\theta) = (1-\theta)F(\theta)$.

Example 4.3 (State-independent Linear Loss)**.** Assume $v(q,\theta) = -|q-1|$, $u(q,\theta) = -(q-\theta)^2$, and $\Theta = [0,1]$. This is equivalent to $v(q,\theta) = (q-1)\cdot \mathrm{sgn}(1-q)$ and $u(q,\theta) = \theta q - q^2/2$. Then, if $\theta_0 \leq 1/2$, $r(\theta) = f(\theta)$ and $R(\theta) = F(\theta)$ because $q_i(\theta_0) = 2\theta_0 \leq 1$. Otherwise, if $\theta_0 \geq 1/2$,

$$r(\theta) = \begin{cases} f(\theta), & \text{if } \theta \leq \theta_0 \\ -f(\theta), & \text{if } \theta \geq \theta_0 \end{cases} \text{ and } R(\theta) = \begin{cases} F(\theta), & \text{if } \theta \leq \theta_0 \\ 2F(\theta_0) - F(\theta), & \text{if } \theta \geq \theta_0 \end{cases}$$

In particular, because $\underline{\theta} = 0$, $r(\theta)$ is quasi-I if and only if $f(\theta)$ is quasi-I for $I \in \{$unimodal, increasing, decreasing$\}$. Thus, the optimal deterministic rating scheme

- is fully revealing if and only if $f(\theta)$ is decreasing,
- is pass/fail if and only if $f(\theta)$ is quasi-increasing, and
- is lower censorship if and only if $f(\theta)$ is quasi-unimodal.

Remark 5. In the two examples above, the results implied by Proposition 3 is consistent with Propositions 1, 2, and 3 in Kartik, Kleiner and Van Weelden (2021), as the triplet (fully revealing test, pass-fail, lower censorship) corresponds to (full delegation, no compromise, interval delegation) in their setting.[54]

[53] To see this, his assumption 3 that $u(q_f(\bar{\theta}),\theta) \geq 0$ for all $\theta \in [\underline{\theta},\bar{\theta}]$ is equivalent to $q_i(\theta) \geq q_f(\bar{\theta})$ for all $\theta \in [\underline{\theta},\bar{\theta}]$, which is equivalent to $q_i(\underline{\theta}) \geq q_f(\bar{\theta})$ and thus $\bar{\theta} \leq \theta_c(\underline{\theta})$.

[54] They also allow for stochastic delegation mechanisms, which are not considered here.

State-Dependent Preferences. Linear delegation also includes cases where the principal's preference depend on the agent's type. In many applications, the principal internalizes a fraction $\alpha \in [0,1]$ of the agent's costs, $v(q,\theta) = q - \alpha c(q)/\theta$. For example, this can arise from parents (who pay tuition fees to the school) partially internalizing the student's cost (Onuchic and Ray, 2023), students overestimating their costs, social spillovers of quality investment, or the regulatory certifier maximizing a weighted sum of firms' profit and average quality (Bizzotto and Harstad, 2023) because $\mathbf{E}[\alpha q(\theta) + (1-\alpha)U(\theta)] = \mathbf{E}[q(\theta) - \alpha c(q(\theta))/\theta]$.

Example 4.4 (Partial Cost Internalization)**.** Assume $v(q,\theta) = q - \alpha c(q)/\theta$. Then,

$$r(\theta) = (1-\alpha)f(\theta) - \alpha(\tilde{F}(\theta) - \tilde{F}(\theta_0)),$$

where $\tilde{F}(\theta) = \int_{\underline{\theta}}^{\theta} f(x)/x \, \mathrm{d}x$.[55] The function $r(\theta)$ is a weight sum of the density $f(\theta)$ and a decreasing function $-(\tilde{F}(\theta) - \tilde{F}(\theta_0))$.

In the utilitarian benchmark where $\alpha = 1$, because $r(\theta) = -(\tilde{F}(\theta) - \tilde{F}(\theta_0))$ is decreasing on $\mathbb{R}_+$, a fully revealing test is optimal. As α decreases to 0, a minimum standard becomes optimal because $r(\theta)$ is no longer decreasing in $\mathbb{R}_+$ (unless $\underline{\theta} = 0$). Intuitively, as the preference misalignment increases, it is optimal to have a minimum standard to provide stronger incentives for high types (possibly at the cost of excluding low types). If $f(\theta)$ is decreasing, the optimal minimum standard will not lead to exclusion because $r(\theta)$ is decreasing on $[\underline{\theta}, \bar{\theta}]$ and thus quasi-decreasing on $\mathbb{R}_+$. On the other hand, if $f(\theta)$ is unimodal or increasing, tests with minimum standard (lower censorship or pass/fail) that entails exclusion can be optimal.

The following example shows that for a particular state-dependent preference, the characterization of the optimal deterministic rating scheme is the same as that for quality maximization.

Example 4.5 (Kolotilin and Zapechelnyuk, 2019, Proposition 1)**.** Assume $v(q,\theta) = \theta q - c(q)/2$. Then, $r(\theta) = \theta f(\theta)/2 - (F(\theta) - F(\theta_0))/2$ is increasing (decreasing) on $[\underline{\theta}, \bar{\theta}]$ if and only if $f(\theta)$ is increasing (decreasing) on $[\underline{\theta}, \bar{\theta}]$.[56] Thus, the same results for quality maximization hold: the optimal deterministic rating is lower censorship (pass/fail) if $f(\theta)$ is unimodal (increasing).

[55] To see this, note that $\int_{\underline{\theta}}^{\bar{\theta}} (q - \alpha c(q)/\theta) \, \mathrm{d}F(\theta) = \int_{\underline{\theta}}^{\bar{\theta}} (\theta q - \alpha c(q)) \, \mathrm{d}\tilde{F}(\theta)$, and that $v(q,\theta) = \theta q - \alpha c(q)$ induces $r(\theta) = (1-\alpha)\theta f(\theta) - \alpha(F(\theta) - F(\theta_0))$.

[56] This is because $r(\theta_0) = \theta_0 f(\theta_0)/2$ and $r'(\theta) = \theta f'(\theta)/2 \overset{\text{sign}}{=} f'(\theta)$.

Next, I consider quadratic loss utility functions (with downward bias), a widely studied case in optimal delegation (see, e.g., Alonso and Matouschek, 2008; Kováč and Mylovanov, 2009; Kleiner et al., 2021).

Example 4.6 (Quadratic Loss)**.** Assume $v(q,\theta) = -(q-\beta(\theta))^2$ and $u(q,\theta) = -(q-\theta)^2$ with $\beta(\theta) \geq \theta$ and $\Theta = [0,1]$. This is equivalent to $v(q,\theta) = \beta(\theta)q - q^2/2$ and $u(q,\theta) = \theta q - q^2/2$ (i.e., linear delegation with $c(q) = q^2/2$ and $\alpha = 1$). Then,

$$r(\theta) = (\beta(\theta) - \theta)f(\theta) - (F(\theta) - F(\theta_0)).$$

In particular, Proposition 3 implies that a fully revealing test is optimal if and only if $r(\theta)$ is decreasing.[57]

4.2.4 General Preferences

In this subsection, I consider the general case where the principal's preference $v(q,\theta)$ only needs to satisfy Assumptions 1 (downward bias) and other mild assumptions: twice continuously differentiable, $v_{qq}(q,\theta) \leq 0$, $v(0,\theta) = 0$, and $v_q(0,\theta) > 0$ for all $q \in Q$ and $\theta \in \Theta$. In particular, it does not necessarily satisfy Condition LD (linear delegation).

For example, the principal may partially internalize (a possibly nonconstant fraction of) the agent's cost—i.e., $v(q,\theta) = \theta q - \tilde{c}(q)$, where $\tilde{c}(q)$ is increasing, convex, and satisfies $\tilde{c}'(q) \leq c'(q)$ for all $q \in Q$. Because the marginal payoffs are $v_q(q,\theta) = \theta - \tilde{c}'(q)$ and $u_q(q,\theta) = \theta - c'(q)$, this belongs to the "nonlinear delegation" case (see Kolotilin and Zapechelnyuk, 2019).

In this case, the characteristic functions $r(\theta)$ and $R(\theta)$ can take more general forms. However, the conditions in Proposition 3 remain *sufficient* conditions for lower censorship and pass/fail tests, with the $r(\theta)$ function in conditions (S) and (C) are replaced by a more complicated function, which incorporates the principal's preferences through the relative concavity $-v_{qq}(q,\theta)/c''(q)$.

Definition of $r(\theta)$ for General Preferences. Define the "relative concavity" of the principal and agent's preferences by

$$\kappa = \inf_{q\in Q,\theta\in[\underline{\theta},\bar{\theta}]} \{-v_{qq}(q,\theta)/c''(q)\}. \tag{15}$$

[57] Cf. Alonso and Matouschek (2008, Proposition 3), as well as Kováč and Mylovanov (2009, Corollary 3) and Kleiner et al. (2021, Corollary 5), who allow for stochastic delegation.

Define

$$r(\theta|q) = v_q(q,\theta)f(\theta) - \kappa(\theta - c'(q))f(\theta) - \kappa(F(\theta) - F(\theta_0)). \tag{16}$$

Recall that $\theta_c(\theta_0) = c'(q_i(\theta_0))$. Slightly abusing the notation, substituting $q = q(\theta)$ of lower censorship or pass/fail in equation (8) into $r(\theta|q)$, define

$$r(\theta) = r(\theta|q(\theta)) = \begin{cases} v_q(0,\theta)f(\theta) - \kappa\theta f(\theta) - \kappa(F(\theta) - F(\theta_0)), & \text{if } \theta \in [\underline{\theta}, \theta_0) \\ v_q(q_i(\theta_0),\theta)f(\theta) - \kappa(\theta - \theta_c(\theta_0))f(\theta) - \kappa(F(\theta) - F(\theta_0)), & \text{if } \theta \in [\theta_0, \theta_c(\theta_0)) \\ v_q(q_f(\theta),\theta)f(\theta) - \kappa(F(\theta) - F(\theta_0)), & \text{if } \theta \in [\theta_c(\theta_0), \bar{\theta}] \end{cases} \tag{17}$$

and $R(\theta) = \int_{\underline{\theta}}^{\theta} r(\tilde{\theta})\,\mathrm{d}\tilde{\theta}$.[58] Note that $r(\theta)$ may be discontinuous at θ_0 because $r(\theta_0^-) \geq r(\theta_0^+)$ (see Lemma B.3).

As before, by convention, $r(\theta) = \kappa F(\theta_0) \geq 0$ for all $\theta < \underline{\theta}$ and $r(\theta) = -\kappa(1 - F(\theta_0)) \leq 0$ for all $\theta > \bar{\theta}$. Under the general definition,

$$\begin{aligned} L(\theta|\theta_0) &= \frac{R(\theta_0) - R(\theta)}{\theta_0 - \theta} \\ &= \begin{cases} \frac{1}{\theta_0 - \theta}\left[\int_{\theta}^{\theta_0} v_q(0,\tilde{\theta})f(\tilde{\theta})\,\mathrm{d}\tilde{\theta} - \kappa\theta(F(\theta_0) - F(\theta))\right], & \text{if } \theta \in [\underline{\theta}, \theta_0), \\ \frac{1}{\theta - \theta_0}\left[\int_{\theta_0}^{\theta} v_q(q_i(\theta_0),\tilde{\theta})f(\tilde{\theta})\,\mathrm{d}\tilde{\theta} - \kappa(\theta - \theta_c(\theta_0))(F(\theta) - F(\theta_0))\right], & \text{if } \theta \in (\theta_0, \theta_c(\theta_0)],\text{[59]} \end{cases} \end{aligned} \tag{18}$$

which is the slope of the line connecting θ_0 and θ on $R(\theta)$. Recall that $A(\theta_0) = L(\theta_c(\theta_0)|\theta_0)$.

Proposition 4. *The optimal deterministic rating scheme*

- *is lower censorship (with cutoff type θ_0^*) if $r(\theta)$ is quasi-unimodal (with conditions (S) and (C) satisfied at θ_0^*),*
- *is pass/fail if $r(\theta)$ is quasi-increasing,*
- *induces no exclusion if $r(\theta)$ is quasi-decreasing, and*
- *is fully revealing if $r(\theta)$ is decreasing on $\mathbb{R}_+$.*

Proof sketch. The proof of sufficiency is the same as the proof of Proposition 3, with $r(\theta)$ function replaced by the more general function in equation (17). □

[58] Note that for all $\theta \geq \bar{\theta}$, we have $r(\theta) = -\kappa(1 - F(\theta_0))$ so that $R(\theta) = R(\bar{\theta}) - \kappa(1 - F(\theta_0))(\theta - \bar{\theta})$ because $f(\theta) = 0$ and $F(\theta) = 1$.

[59] This follows from applying exchanges of integrals to $\int_{\theta_0}^{\theta} r(\tilde{\theta})d\tilde{\theta}$.

4.3 Relationship to the Literature

Relationship to the Delegation Literature. As mentioned above, the triplet (fully revealing test, pass-fail, lower censorship) corresponds to (full delegation, take-it-or-leave-it offers, threshold delegation) in delegation. Amador and Bagwell (2022) study the problem of regulating a monopolist without transfers and characterize sufficient conditions for threshold delegation (i.e., price-cap regulation) to be optimal. I improve upon their results and obtain weaker sufficient conditions that are also necessary in the linear delegation case (see Propositions 3 and 4). Importantly, my conditions allow for the bang-bang allocation induced by pass/fail tests (take-it-or-leave-it delegation) and exclusion in lower censorship (price-cap allocation). By contrast, their conditions imply a fully revealing region and need to hold at *all* possible cutoff types $\theta_0 \in [\underline{\theta}, \bar{\theta})$, thereby ruling out the optimality of the bang-bang allocation and exclusion (in the absence of a fixed production cost).

The contribution is meaningful in two ways. On the one hand, pass/fail tests (bang-bang allocation) and exclusion are ubiquitous in practice. In their setting, the bang-bang allocation, where the firm either sets the price at the cap or shuts down, is also common.[60] On the other hand, my sufficient conditions in Corollary 3.1 are also easy to check (because it does not require one to find the θ_0) and still weaker than their conditions. For example, in the quality maximization case (or $v(q, \theta) = \theta q - c(q)/2$), their conditions for lower censorship require a quasi-decreasing density, while mine also include (quasi-)unimodal and (quasi-)increasing densities. See Appendix C for the comparison and technical details.

In contrast to expertise-based delegation, Kartik, Kleiner and Van Weelden (2021) study delegation in veto-bargaining and provide necessary and sufficient conditions for the optimality of interval delegation, full delegation, and no compromise (which corresponds to the pass/fail test). My results in this special case are consistent with theirs (see Examples 4.2 and 4.3). However, they assume a specific state-independent principal preference (with a constant bliss point) and a quadratic agent preference. They also cover stochastic delegation mechanisms, but they are different from the stochastic ratings in this paper.

Relationship to the Persuasion Literature. In the linear delegation case, Kolotilin and Zapechelnyuk (2019) translate it to an equivalent linear persuasion problem and solve it using methods in Kolotilin (2018) and Dworczak and Martini (2019). My method uses

[60] In their context, take-it-or-leave-it offers take the form of fixed-price regulation where the firm either accepts the fixed price or shuts down.

the optimal control approach that parallels with the Lagrangian approach developed by Amador, Werning and Angeletos (2006), which allows for *nonlinear* delegation, where the principal's marginal payoff is nonlinear in (a transformation of) the agent's action. In this case, in the equivalent persuasion problem, the sender's payoff is nonlinear in the state (so that not only the posterior mean matters), so their methods no longer apply.

Relationship to Kleiner et al. (2021). In Section 4.3 of Kleiner, Moldovanu and Strack (2021), they study optimal delegation with potentially stochastic mechanisms (and quadratic utilities) as an application of their results on extreme points and majorization.[61] For example, in the quality maximization case $v(q,\theta)=q$, if $c(q,\theta)=q^2/2\theta$, an undominated deterministic allocation $q(\theta)$ is incentive-compatible if and only if there exists an extension $\tilde{q}(\theta)$ of $q(\theta)$ to $[\underline{\theta},\bar{\theta}']$ such that $\tilde{q}(\bar{\theta}')=\bar{\theta}'$ and $\tilde{q}(\theta)$ is an extreme point of the mean-preserving contraction (MPC) of

$$q_c(\theta)=\begin{cases} q_i(\underline{\theta})=2\underline{\theta}, & \text{if } \theta\in[\underline{\theta},\theta_L(\underline{\theta})] \\ q_f(\theta)=\theta, & \text{if } \theta\geq\theta_L(\underline{\theta}). \end{cases}$$

Therefore, the optimality of pass/fail tests (lower censorship) in quality maximization when the density is increasing (unimodal) directly follows from their Proposition 3.[62]

4.4 Beyond Lower Censorship

In this subsection, I characterize the optimal deterministic rating schemes beyond lower censorship. By the revelation principle and Lemma 2, the optimal deterministic rating can be characterized by the quality scheme it induces. By Lemma 4, the induced quality scheme consists of pooling and fully revealing intervals and at most countably many jump discontinuities. Thus, I label the exclusion interval as $[\underline{\theta},\theta_0]$, and other pooling and fully revealing intervals as $[\theta_0,\theta_1],\ldots,[\theta_{k-1},\theta_k]$, where $k\geq 1$.[63] Denote $q_j=q(\theta_j+)$ for all $j\geq 0$. As a convention, denote $q_{-1}=0$ and $\theta_{-1}=\underline{\theta}$. Thus, $(q_{-1},q_0,q_1,\ldots,q_{k-1})$ is an increasing sequence.

[61]Their approach can be extended to more general preferences, such as the linear delegation case.

[62]They also extend the equivalence between delegation and persuasion established by Kolotilin and Zapechelnyuk (2019) to delegation with stochastic mechanisms and persuasion with general information structures. However, the rating design problem I consider differs from the Bayesian persuasion setting in Kolotilin and Zapechelnyuk (2019) where the principal designs the information sent to the agent about the state (i.e., type). Thus, the rating design problem with stochastic ratings in Section 5 is *not* equivalent to stochastic delegation (or persuasion).

[63]The labeling is possible because $q(\theta)$ has at most countably many jumps.

For any two adjacent pooling intervals $[\theta_{j-1}, \theta_j]$ (on which $q(\theta) = q_{j-1}$) and $[\theta_j, \theta_{j+1}]$ (on which $q(\theta) = q_j$), Lemma 4 implies $q_{j-1} - c(q_{j-1})/\theta_j = q_j - c(q_j)/\theta_j$ at the jump θ_j. Each jump at θ_j corresponds to a minimum standard $q_j > 0$ in the rating scheme π.

If a pooling interval is adjacent to a fully revealing interval, $q(\theta)$ must be continuous at the boundary θ_j of the pooling interval, i.e., $q_j = q_f(\theta_j)$ on the pooling interval.

Example (Lower censorship)**.** Lower censorship is a special case of $k \leq 2$. When $k = 2$, $[\underline{\theta}, \theta_0]$ and $[\theta_0, \theta_1]$ are the pooling intervals, and $[\theta_1, \bar{\theta}]$ is the fully revealing interval; q_0 is the only minimum standard. At $\theta_0 > \underline{\theta}$, $q_0 - c(q_0)/\theta_0 = 0$ (i.e. $q_0 = q_i(\theta_0)$); at $\theta_1 \leq \bar{\theta}$, $q_0 = q_f(\theta_1)$ (i.e. $\theta_1 = \theta_L(\theta_0)$).

Example (Two minimum standards)**.** Assume $c(q) = q^2/2$ and $\Theta = [0, 5]$. Then, an incentive-compatible quality scheme is $q(\theta) = \begin{cases} 0; & \theta \in [0,1] \\ 2; & \theta \in [1,3] \\ 4; & \theta \in [3,4] \\ \theta; & \theta \in [4,5] \end{cases}$, which has two jumps (see Figure 10), so the rating scheme has two minimum standards $q_0 = 2$ and $q_1 = 4$.

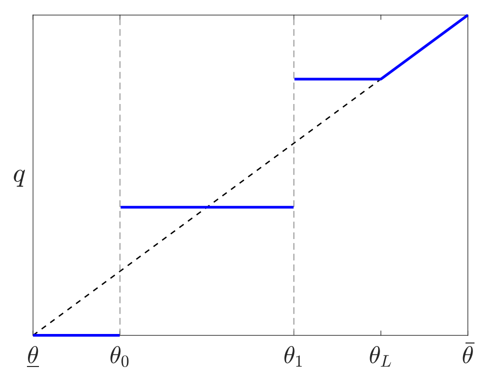


Figure 10: Two minimum standards induce $q(\theta)$ with two jumps

Beyond lower censorship, the optimal control method can still be applied to solve for the optimal deterministic rating in general. On the technical side, the Hamiltonian multiplier $\mu = 0$ must hold at any $\theta_j \in (\underline{\theta}, \bar{\theta})$ and on any fully revealing interval. Moreover, the switching condition needs to hold at any $\theta_j \in (\underline{\theta}, \bar{\theta})$ at which $q(\theta)$ is discontinuous.

Under Condition LD, define the characteristic function by

$$r_j(\theta) = (\beta(\theta) - \alpha\theta)f(\theta) - \alpha(F(\theta) - F(\theta_j)) \tag{19}$$

Analogously, define $R_j(\theta) = \int_{\underline{\theta}}^{\theta} r_j(\tilde{\theta})\,\mathrm{d}\tilde{\theta}$, $L_j(\theta|\theta_j) = \frac{R(\theta_c(\theta_j)) - R(\theta_j)}{\theta_c(\theta_j) - \theta_j}$, and $A_j = L_j(\theta_c(\theta_j)|\theta_j)$.

I state the following sufficient conditions on the pooling and fully revealing intervals for the optimal deterministic rating scheme, as extensions of conditions (S) and (C).

Condition (S-j). On any two adjacent pooling intervals $[\theta_{j-1}, \theta_j]$ (where $q(\theta) = q_{j-1}$) and $[\theta_j, \theta_{j+1}]$ (where $q(\theta) = q_j$),

$$\int_{\theta_j}^{\theta} r_j(\tilde{\theta}) d\tilde{\theta} \geq A_j \cdot (\theta - \theta_j) \text{ for all } \theta \in [c'(q_{j-1}), c'(q_j)],$$

with equality if $\theta \in \{c'(q_{j-1}), c'(q_j)\} \cap (\underline{\theta}, \bar{\theta})$.[64]

Condition (C-j). On any fully revealing interval $[\theta_j, \theta_{j+1}]$, $r_j(\theta)$ is decreasing in θ.

I propose the following sufficient conditions on the pooling and fully revealing intervals for the optimal deterministic rating scheme, as extensions of conditions (S) and (C).

Proposition 5. *The quality scheme is optimal if conditions* (S-j) *and* (C-j) *holds on all pooling and fully revealing intervals, respectively*

Analogous to the case of lower censorship, the sufficient conditions (S-j) and (C-j) are related to the modes of the density $r(\theta)$.

Corollary 5.1. *If $r(\theta)$ has $n \geq 1$ modes, the optimal deterministic rating scheme has at most n minimum standards.*[65] *If the smallest mode is in the interior of $[\underline{\theta}, \bar{\theta}]$, the optimal deterministic rating scheme must have a minimum standard at the bottom (below which a "fail" signal is disclosed).*

Example (Quality Maximization)**.** Assume $v(q, \theta) = q$ (or $v(q, \theta) = \theta q - c(q)/2$). If $f(\theta)$ is bimodal,[66] the optimal deterministic rating can have at most two minimum standards, that is, high-pass/low-pass/fail.

5 Optimal General (Stochastic) Ratings

5.1 Principal's Problem

In this section, I study the optimal rating design without the restriction to deterministic rating schemes. In other words, $w(\theta) = q(\theta)$ is no longer necessary, and the constraints

[64] Recall the convention that $q_{-1} = 0$ and $\theta_{-1} = \underline{\theta}$

[65] When f is constant in some regions, there are potentially many optimal deterministic rating schemes (or $q(\theta)$), and I consider the one with the fewest minimum standards (or jumps).

[66] For example, the ability distribution in squadrons in Carrell et al. (2013) is bimodal.

(MPS) and (BP) must hold instead. With a multiplicatively separable cost function $c(q)/\theta$, the principal's problem [P] is

$$[\text{P}] \qquad \max_{q(\theta),w(\theta)} \int_{\underline{\theta}}^{\bar{\theta}} v(q(\theta),\theta)\,\mathrm{d}F(\theta) \tag{20}$$

subject to, for all $\theta \in [\underline{\theta},\bar{\theta}]$,

$$q(\theta) \text{ increasing} \qquad \text{(IC-Mon)}$$

$$\theta w(\theta) - c(q(\theta)) = \int_{\underline{\theta}}^{\theta} w(x)dx + \underline{U} \qquad \text{(IC-Env)}$$

$$\theta w(\theta) - c(q(\theta)) \geq 0 \qquad \text{(IR)}$$

$$\int_{\underline{\theta}}^{\theta} w(\theta')\,\mathrm{d}F(\theta') \geq \int_{\underline{\theta}}^{\theta} q(\theta')\,\mathrm{d}F(\theta'), \qquad \text{(MPS)}$$

$$\int_{\underline{\theta}}^{\bar{\theta}} w(\theta)\,\mathrm{d}F(\theta) = \int_{\underline{\theta}}^{\bar{\theta}} q(\theta)\,\mathrm{d}F(\theta) \qquad \text{(BP)}$$

Define $D(\theta) = \int_{\underline{\theta}}^{\theta}(w(\theta') - q(\theta'))\,\mathrm{d}F(\theta')$. In the optimal control problem, I add $D(\theta)$ as a state variable subject to $D(\theta) \geq 0$ (MPS) with the Lagrangian multiplier $\lambda(\theta) \geq 0$ and $\dot{D} = [w(\theta) - q(\theta)]f(\theta)$ with the Hamiltonian multiplier $\Lambda(\theta)$. By construction, $\dot{\Lambda}(\theta) = -\lambda(\theta) \leq 0$. The complementary-slackness conditions on (MPS) are $\lambda(\theta)D(\theta) = 0$ and $\lambda(\theta) \geq 0$. See Appendix A.2 for details.

5.2 When are deterministic ratings optimal?

The optimal rating scheme is deterministic if and only if (MPS) holds with equality for all θ. Thus, we can use a guess-and-verify approach by plugging in the quality scheme induced by the optimal deterministic rating scheme. As long as the resulting solution satisfies the complementary-slackness condition $\lambda(\theta) \geq 0$ on (MPS), the optimal deterministic rating scheme remains optimal among general rating schemes.

5.2.1 Quality Maximization

I start with the case where the principal maximizes expected quality, i.e., $v(q,\theta) = q$.

Proposition 6. *Assume $v(q,\theta) = q$. The optimal rating scheme is deterministic if $f(\theta)$ is increasing in the exclusion and bunching regions of the optimal deterministic rating* *scheme, while $\theta f'(\theta)/f(\theta)$ (and $f(\theta)$) are decreasing in the fully revealing region of the optimal deterministic rating scheme.*

Remark 6. $\varepsilon_f(\theta) \equiv \theta f'(\theta)/f(\theta)$ is the *elasticity* of $f(\theta)$.[67] It is decreasing if and only if $f''(\theta) \leq f'(\theta)^2/f(\theta) - f'(\theta)/\theta$, which relates to the relative concavity of $F(\theta)$ and $f(\theta)$.[68]

Remark 7. $\varepsilon_f(\theta)$ is decreasing for most well-known unimodal distributions in the decreasing region of $f(\theta)$, including uniform, exponential, Pareto, log-normal, and normal distributions, even for distributions that violate monotone hazard rate property and Myerson's regularity (e.g., log-normal and Pareto).

The Pareto distribution Par(α, β) has a strictly decreasing density $f(\theta) = \alpha\beta^\alpha\theta^{-(\alpha+1)}$ and a *constant* elasticity $\theta f'(\theta)/f(\theta) = -(\alpha + 1)$. The condition can be violated by distributions that have heavier tails than Pareto, such as certain Type-II (Fréchet-type) generalized extreme value distributions.

The proposition implies that the optimality of pass/fail and fully revealing tests is somewhat robust to stochastic rating schemes.

Corollary 6.1. *Assume $v(q, \theta) = q$. The optimal rating scheme is pass/fail if $f(\theta)$ is increasing.*[69] *The optimal rating scheme is fully revealing if and only if both $f(\theta)$ and $\theta f'(\theta)/f(\theta)$ are decreasing and $\underline{\theta} = 0$.*

The corollary implies that pass/fail tests remain optimal if types are increasingly more concentrated towards the top of the distribution. A fully revealing test is optimal if types are increasingly more concentrated towards the bottom of the distribution and the elasticity of density is also decreasing, which suggests a relatively thin tail, (and $\underline{\theta} = 0$).

It is worth noting that lower censorship with nonempty exclusion and fully revealing regions (e.g., when the density is unimodal) is unlikely to be optimal among stochastic ratings. To see this, recall from the previous section that the mode of the unimodal $f(\theta)$ is in the bunching region $(\theta_0^*, \theta_L(\theta_0^*))$ of the optimal quality scheme. But for it to remain optimal, the density $f(\theta)$ needs to be increasing on the entire bunching region. Thus, a unimodal density will not satisfy this condition in general.

[67] Consider $1 - F(\theta)$ as the demand function. Then, $\theta f'(\theta)/f(\theta)$ represents the curvature (or convexity) of the demand function, or the elasticity of the slope of demand, in the industrial organization literature (see, e.g., Seade (1980); Aguirre et al. (2010); Mrázová and Neary (2017)).

[68] If $f(\theta)$ is strictly decreasing, the condition becomes $f''(\theta)/f'(\theta) \geq f'(\theta)/f(\theta) - 1/\theta$

[69] Another sufficient condition is both $\theta_0^* = \underline{\theta}$ (i.e., no exclusion) and $\bar{\theta} \leq \theta_c(\underline{\theta})$ (i.e., variation in types is not large enough to sustain a fully revealing region).

5.2.2 Linear and Nonlinear Delegation

In general, we have the following results in the fully revealing region. Define

$$N_1(\theta) = \left(\frac{v_{qq}(q_f(\theta), \theta)}{c''(q_f(\theta))} + v_{q\theta}(q_f(\theta), \theta) \right) \theta + v_q(q_f(\theta), \theta) \left(1 + \frac{\theta f'(\theta)}{f(\theta)} \right). \tag{21}$$

Example (Linear Delegation)**.** Under Condition LD, $N_1(\theta) = (\beta'(\theta) - \alpha)\theta + (\beta(\theta) - \alpha\theta)[1 + \theta f'(\theta)/f(\theta)]$. When $v(q, \theta) = q$, $N_1(\theta) = 1 + \theta f'(\theta)/f(\theta)$.

Condition (N1). $N_1(\theta)$ is decreasing in θ.

Lemma 7. *If the optimal deterministic rating scheme fully reveals $\theta \in [\theta_j, \theta_{j+1}]$, then the optimal rating scheme also fully reveals $\theta \in [\theta_j, \theta_{j+1}]$ if and only if $N_1(\theta)$ is decreasing on $[\theta_j, \theta_{j+1}]$.*

Because it provides a necessary and sufficient condition, the lemma also implies that if the optimal deterministic rating scheme has a fully revealing region where $N_1(\theta)$ is not decreasing, then a stochastic rating scheme can strictly improve upon it (see Proposition 8).

In the pooling regions $[\theta_{j-1}, \theta_j]$ and $[\theta_j, \theta_{j+1}]$, the following condition needs to hold.

Condition (N2). $N_2(\theta) = A_j/f(\theta) + \kappa\theta + \kappa(F(\theta) - F(\theta_j))/f(\theta)$ is decreasing in θ.

In particular, for lower censorship or pass/fail tests, $\theta_j = \theta_0$ and $A_j = A(\theta_0)$ as defined in equation (11).[70]

The following proposition provides sufficient conditions for the optimal rating scheme to be deterministic.

Proposition 7. *The optimal rating scheme is deterministic if the optimal deterministic rating scheme satisfies (N1) and (N2) in the fully revealing and pooling regions, respectively.*

The optimal rating scheme is a pass/fail test if the optimal deterministic rating scheme is pass/fail and condition (N2) *holds on* $[\underline{\theta}, \bar{\theta}]$.

Proof sketch. Use the same multipliers as in the deterministic ratings (where $D(\theta) \equiv 0$). Conditions (N1) and (N2) guarantee the Lagrangian multiplier on $D \geq 0$ is positive. $\square$

[70] If $(\theta_0, \theta_L(\theta_0)) = (\underline{\theta}, \bar{\theta})$ (i.e., bunching without exclusion), the condition can be relaxed to $\tilde{N}_2(\theta) = \kappa\theta + \kappa F(\theta)/f(\theta)$ is decreasing.

5.3 When are stochastic ratings optimal?

Since stochastic ratings expand the set of incentive-compatible quality $q(\theta)$, a natural question is when stochastic ratings are optimal.

By Lemma 7, whenever condition (N1) does not hold in the fully revealing region of the deterministic rating, stochastic ratings can improve upon deterministic ratings. When $v(q,\theta) = q$, this means that if the optimal deterministic rating has a fully revealing region in which the elasticity of density, $\theta f'(\theta)/f(\theta)$, is not decreasing, then it is not optimal among general (possibly stochastic) rating schemes.

Proposition 8. *Stochastic rating schemes can strictly improve on deterministic rating schemes if the optimal deterministic rating has a fully revealing region in which Condition* (N1) *does not hold.*

Proof. Follows immediately from Lemma 7. □

Intuition. Intuitively, if $q'(\theta) > 0$, one can write $c'(q(\theta))/\theta = \hat{w}'(q(\theta)) \equiv w'(\theta)/q'(\theta)$. Stochastic rating schemes can allow $\hat{w}'(q) > 1$ for some qualities to provide stronger marginal incentives than fully revealing the marginal investment in quality to the market (i.e., $\hat{w}(q) = q$). This can be achieved, for example, by increasing the probability of the agent's quality being pooled with higher qualities (or separated from lower qualities). Consequently, this partial pooling leads to higher $q(\theta)$ for some (lower) types at the expense of lower $q(\theta)$ for other (higher) types, which can be more desirable for the principal under some distributions.[71]

Example 5.1. Assume $v(q,\theta) = q$ and $\Theta = [0,1]$. If $f(\theta)$ is decreasing but the elasticity $\varepsilon_f(\theta) = \theta f'(\theta)/f(\theta)$ is not decreasing (e.g., distributions with heavier tails than Pareto), then a stochastic rating scheme can strictly improve upon the optimal deterministic rating scheme (which is fully revealing).

Intuitively, for a decreasing density, its elasticity $\varepsilon_f(\theta)$ is not decreasing if it has a very "fat tail"—that is, there are a few very high types at the top of the distribution. In this case, it can be beneficial to partially pool low types with high types to induce higher quality from low types at the cost of incentives for high types.

Similarly, in the pooling region, by partially pooling low types with high types, stochastic rating schemes allow the principal to set a higher minimum standard without discouraging participation, in contrast to deterministic rating schemes where a higher minimum

[71] If Assumption 1 (downward bias) is violated (i.e., $q_e(\theta) < q_f(\theta)$), that is, if the principal wants to induce a lower quality than the agent would choose under full revelation, stochastic ratings can lead to a more flexible $q(\theta)$ which can be closer to the principal-optimal $q_e(\theta)$.

standard leads to more exclusion. Thus, stochastic rating schemes can increase the incentives for low types at the cost of incentives for higher types, thereby potentially improve on deterministic schemes.

Example 5.2. Assume $v(q,\theta) = q$, $c(q) = q^2/2$. Assume $\Theta = [1, 6]$ and f is decreasing. Then, the optimal deterministic rating is lower censorship without exclusion and with a minimum standard $q_0 = 2$, but it is not optimal among general ratings. Here is a stochastic rating $\pi(q)$ that improves upon it. Fix $q_1 > q_0$. Consider the following noisy test: if $q \geq q_0'$

$$\pi(q) = \begin{cases} q, & \text{with probability } 1 - p(q) \\ q_1, & \text{with probability } p(q) \end{cases} \tag{22}$$

Otherwise, $\pi(q) = \text{fail}$ for all $q < q_0'$.

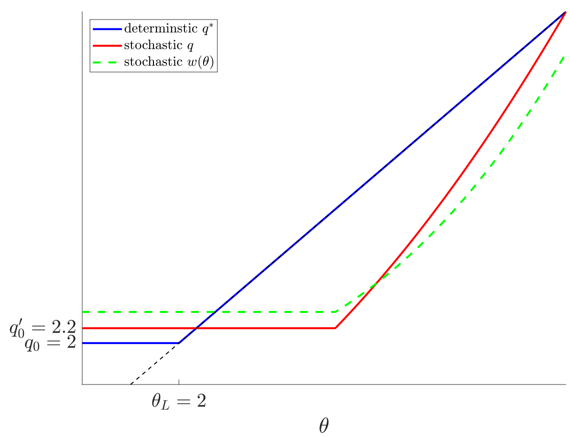


Figure 11: A noisy test can potentially improve upon deterministic ratings

By partially pooling low types with high types, the principal can set a higher minimum standard $q_0' > q_0$, without leading to exclusion, at the cost of incentives for higher types, which could benefit the principal for some distributions with decreasing densities. See Figure 11 for an illustration.

Although I provide sufficient conditions under which stochastic rating schemes can or cannot strictly improve upon deterministic rating schemes, solving for the optimal rating scheme is more challenging. Because the rating is deterministic if and only if $w(\theta) = q(\theta)$, the optimal rating is stochastic on some interval $[\theta_1, \theta_2]$ if and only if $D(\theta) = \int_{\underline{\theta}}^{\theta} (w(x) - q(x)) \, \mathrm{d}F(x) > 0$ on (θ_1, θ_2), which by the complementary-slackness conditions, holds only if $\lambda(\theta) = 0$ (i.e., (MPS) does not bind).[72] The example below shows for quality

[72] In particular, by Corollary 1.1, if $c_q(q(\theta), \theta) \leq 1$ for all $\theta \in [\underline{\theta}, \bar{\theta}]$, then (BP) implies (MPS), and therefore (MPS) does not bind on $[\underline{\theta}, \bar{\theta}]$.

maximization, (MPS) has to be binding for some types, so the optimal rating must be deterministic in those regions.

Example (Running example: linear-quadratic)**.** Assume $v(q,\theta) = q$ and $c(q,\theta) = q^2/2\theta$. $\theta \sim \text{Unif}[1,5]$. If (MPS) does not bind,

$$c_q(q(\theta),\theta) = 2 + \frac{1-F(\theta)}{f(\theta)} c_{\theta q}(q(\theta),\theta), \text{ which is} > 1 \text{ for large } \theta.$$

The solution $q(\theta) = 0.4\theta^2$ and $w(\theta) = \frac{8}{75}\theta^3 - \frac{2}{75}$ violate (MPS). Therefore, (MPS) must be binding in some regions, where the optimal rating must be deterministic.

5.4 With a Constant Testing Fee

To show further cases where stochastic tests can be optimal, I introduce a constant testing fee $P > 0$ from the agent to the principal if he takes the test (see Appendix E.3 for details).

With a testing fee, Lemma 1 still holds: an agent takes the test if and only if he chooses $q > 0$; if he does not, the market offers $w_\varnothing = 0$. Denote $\mathbf{1}[q] = \mathbf{1}[q > 0]$. The principal is also concerned with expected transfer $\mathbf{E}[P \cdot \mathbf{1}[q]]$ in addition to $v(q,\theta)$.

Example 5.3 (Optimal certification by a monopoly certifier)**.** In Albano and Lizzeri (2001) with a constant certification fee P, the agent's utility is $U(\theta) = w(\theta) - c(q(\theta),\theta) - P \cdot \mathbf{1}[q(\theta)]$. The principal maximizes expected certification fee $\mathbf{E}[P \cdot \mathbf{1}[q(\theta)]]$, so $v_q(q,\theta) = 0$. Thus, ignoring (MPS), we have

$$c_q(q(\theta),\theta) = 1 + \frac{1-F(\theta)}{f(\theta)} c_{\theta q}(q(\theta),\theta) < 1,$$

Thus, (MPS) is redundant, and the optimal rating is a noisy test because $w'(\theta)/q'(\theta) = c_q(q(\theta),\theta) < 1$.[73]

Example 5.4 (Optimal certification by a regulator)**.** Suppose instead the principal is a benevolent certifier who maximizes a weighted sum of the certification fee and the firm's (agent's) profit, that is, $\mathbf{E}[\alpha P \cdot \mathbf{1}[q(\theta)] + (1-\alpha)U(\theta)]$. Ignoring (MPS), we have

$$c_q(q(\theta),\theta) = 1 + \frac{2\alpha - 1}{\alpha} \cdot \frac{1-F(\theta)}{f(\theta)} c_{\theta q}(q(\theta),\theta)$$

When $\alpha \geq 1/2$, $c_q(q(\theta),\theta) \leq 1$ always holds, so (MPS) does not bind, and the noisy test is optimal. When $\alpha < 1/2$ is small (i.e., the certifier cares more about the firm), (MPS)

[73] See also Saeedi and Shourideh (2020) and Xiao (2023b).

becomes binding.

6 Ability Signaling

6.1 Setup

In this section, I consider the alternative case where the market only values the agent's exogenous ability, θ, à la Spence's (1973) signaling model.[74] In this case, the interim wage is $\hat{w}(q) = \mathbf{E}_{s\sim\pi(q)}[\mathbf{E}[\theta|s]]$. As before, denote $w(\theta) = \hat{w}(q(\theta))$.

The lemmas for the equivalence to reduced-form direct mechanism and incentive compatibility still hold. As for feasibility, a theorem similar to Proposition 1 holds.

Theorem 9. *An incentive-compatible direct mechanism $(q(\theta), w(\theta))$ is feasible if and only $w(\theta)$ is a mean-preserving spread of θ in the quantile space, that is,*

(i) $\int_{\underline{\theta}}^{\theta} w(\theta')\,\mathrm{d}F(\theta') \geq \int_{\underline{\theta}}^{\theta} \theta'\,\mathrm{d}F(\theta')$ *for all* $\theta \in [\underline{\theta}, \bar{\theta}]$ *(MPS'),*

(ii) $\int_{\underline{\theta}}^{\bar{\theta}} w(\theta)\,\mathrm{d}F(\theta) = \int_{\underline{\theta}}^{\bar{\theta}} \theta\,\mathrm{d}F(\theta)$ *(BP').*

The theorem can be proven similarly to Proposition 1 à la the proof of Border's theorem in Kleiner, Moldovanu and Strack (2021, Theorem 3). Within deterministic rating schemes, it can only be an extreme point the mean-preserving spread of θ in the quantile space, which is referred to as a "truthful filter" in Rayo (2013). A sufficient condition for feasibility is given below.

Corollary 9.1. *If the incentive-compatible direct mechanism $(q(\theta), w(\theta))$ satisfies $w'(\theta) \leq 1$ on $[\underline{\theta}, \bar{\theta}]$, then it is feasible if and only if it satisfies (BP').*

Because the type θ is exogenous, the rating design problem is simpler than the case where the market values the endogenous quality. On the technical side, because (MPS') and (BP') do not involve the state variable $q(\theta)$, the Hamiltonian becomes simpler as it does not involve pure state constraint. Hence, in this section, I look for the optimal general (possibly stochastic) ratings directly, without having to start by restricting to deterministic ratings.

[74] In the employer example, it is similar to Holmström's (1999) career concern model, except that agents know their abilities.

6.2 Optimal Ratings without Transfers

As in the previous sections, I start by assuming there are no transfers between the principal and the agents. Because the test is costless and always gives a result, taking the test is a strictly dominant strategy for every agent (except the lowest type $\underline{\theta}$ who can be indifferent), even if he invests no effort (i.e., $c(q,\theta)=0$). Therefore, every agent participates in the test, even if he invests no effort, in contrast to the productive investment case. Consequently, $w_\varnothing = \underline{\theta}$.

Lemma 8. *In any equilibrium, if an agent does not take the test, he must be the lowest type $\theta = \underline{\theta}$ who chooses $\underline{q}$ such that $c(\underline{q},\underline{\theta}) = 0$, and the market offers him $w_\varnothing = \underline{\theta}$.*

Assume the cost is multiplicatively separable (see Appendix F.3 for general cost functions). The principal's problem is

$$\max_{q(\theta),w(\theta)} \int_{\underline{\theta}}^{\bar{\theta}} v(q(\theta),\theta)\,\mathrm{d}F(\theta) \tag{23}$$

subject to (MPS'), (BP'), and

$$\theta w(\theta) - c(q(\theta)) = \int_{\underline{\theta}}^{\theta} w(x)\,\mathrm{d}x + \underline{U}, \qquad \text{(IC-Env)} \tag{24}$$

$$q(\theta) \text{ increasing.} \qquad \text{(IC-Mon)} \tag{25}$$

$$\theta w(\theta) - c(q(\theta)) \geq \underline{\theta}\cdot\theta, \qquad \text{(IR)} \tag{26}$$

Say a rating induces *full separation* if $w(\theta)=\theta$.[75] Define $q_f(\theta)$ as the quality scheme under full separation, which is characterized by

$$\hat{w}(q_f(\theta)) \equiv w(\theta) = \theta, \tag{BP}$$

$$q_f(\theta) = \arg\max_q \{\theta\hat{w}(q) - c(q)\} \iff \hat{w}'(q_f(\theta)) = c'(q_f(\theta))/\theta, \tag{FOC}$$

$$\underline{\theta} - c(q_f(\underline{\theta}))/\underline{\theta} = 0. \tag{IR}$$

The first two conditions imply

$$c'(q_f(\theta))\cdot q_f'(\theta) = \theta, \tag{27}$$

which, along with the initial condition in (IR), determines $q_f(\theta)$.

[75] Cf. the fully revealing test in previous sections that induces $\hat{w}(q) = q$ $(w(\theta) = q(\theta))$ when the market values quality.

I also maintain Assumption 1 (downward bias) that $y_q(q_f(\theta),\theta) \geq 0$. Denote

$$J(\theta|q_f) = \frac{y_q(q_f(\theta),\theta)}{c'(q_f(\theta))}\theta - \frac{\int_\theta^{\bar{\theta}} y_q(q_f(x),x)/c'(q_f(x))\,\mathrm{d}F(x)}{f(\theta)} \tag{28}$$

In the linear delegation case where $v(q,\theta) = \beta(\theta)q - \alpha c(q) + d(\theta)$, the expression simplifies to

$$J(\theta|q_f) = \frac{\beta(\theta)}{c'(q_f(\theta))}\theta - \frac{\int_\theta^{\bar{\theta}} \beta(x)/c'(q_f(x))\,\mathrm{d}F(x)}{f(\theta)} - \alpha\left(\theta - \frac{1-F(\theta)}{f(\theta)}\right) \tag{29}$$

If the principal maximizes expected quality—i.e., $v(q,\theta) = q$, then $J(\theta|q_f) = \frac{\theta}{c'(q_f(\theta))} - \frac{\int_\theta^{\bar{\theta}} 1/c'(q_f(x))\,\mathrm{d}F(x)}{f(\theta)}$.

Proposition 10. *The optimal rating scheme induces full separation (i.e., $q^*(\theta) = q_f(\theta)$) if and only if $J(\theta|q_f)$ is increasing in θ.*

Proof sketch. Rewrite the constraints and apply the optimal control methods to the principal's maximization problem. See Appendix A.3 for details. □

Remark 8. For $v(q,\theta) = q$, the result is consistent with Rayo (2013) (which assumes $c(q) = q$) and Zubrickas (2015, Propositions 2 and 3) but does not restrict attention to deterministic ratings.

The necessary and sufficient condition regarding $J(\theta|q_f)$ is reminiscent of that for the optimality of winner-take-all contests in Zhang (2024). Indeed, effort maximization in the ability signaling model is similar to that in contests.

Proposition 10 provides a regularity condition that is necessary and sufficient for full separation to be optimal. In particular, if $v(q,\theta) = q$ and $c(q) = q$, full separation is optimal if and only if $J(\theta) \equiv \theta - \frac{1-F(\theta)}{f(\theta)}$ is increasing.[76]

Example 6.1. Assume $v(q,\theta) = q$, $c(q) = q$ (as in Rayo (2013)), and $\underline{\theta} = 0$. Then $\hat{w}(q) = \sqrt{2q}$ and $q_f(\theta) = \theta^2/2$. The optimal rating induces full separation $q^*(\theta) = \theta^2/2$ if and only if $J(\theta|q_f) = \theta - \frac{1-F(\theta)}{f(\theta)}$ is increasing.

Example 6.2. Assume $v(q,\theta) = q$, $c(q) = q^2/2$, and $\underline{\theta} = 0$. Then, $\hat{w}(q) = q$ and $q_f(\theta) = \theta$. The optimal rating induces full separation $q^*(\theta) = \theta$ if and only if $J(\theta|q_f) = 1 - \frac{\int_\theta^{\bar{\theta}} 1/x\,\mathrm{d}F(x)}{f(\theta)}$ is increasing.

[76] In the quality maximization case with linear cost, Kleiner et al. (2021, Proposition 2) implies that optimal rating scheme is always deterministic because the maximum of a linear function is always obtained at an extreme point.

The following corollary implies that in the quality maximization case, the optimal rating induces full separation at the top under some conditions.

Corollary 10.1 (Cf. Zubrickas, 2015, Propositions 2)**.** *Assume* $v(q,\theta) = q$. *If* $c'(q_f(\theta))/\theta$ *is decreasing in* θ *(or equivalently,* $q_f(\theta)$ *is convex) on* $[\theta_L, \bar{\theta}]$ *for sufficiently large* $\theta_L < \bar{\theta}$*, then the optimal rating induces full separation on* $[\theta_L, \bar{\theta}]$.

6.3 Optimal Ratings with a Constant Testing Fee

In this section, I assume there is a constant testing fee $P \geq 0$ from the agents who take the test to the principal. With the testing fee, the agent of type $\theta > \underline{\theta}$ may not take the test. If so, he will invest no effort but is still endowed with some ability $\theta > \underline{\theta}$. As is the natural of signaling games, multiple equilibria can arise, in contrast to the productive investment case. I will focus on the principal-optimal sequential equilibrium.

Denote the decision dummy by $\sigma(\theta) = \mathbf{1}[\text{type } \theta \text{ takes the test}]$.

Lemma 9. *There exists a cutoff type* θ_0 *such that* $\sigma(\theta) = 1$ *if and only if* $\theta \geq \theta_0$. *Consequently,* $w_\varnothing = \mathbf{E}[\theta \mid \theta \leq \theta_0]$

Assume the principal maximizes a weighted sum of the certification fee and the expected quality, that is, $\mathbf{E}[\sigma(\theta)(\rho P + (1-\rho)q(\theta))]$ where $\rho \in [0,1]$. The principal's problem is

$$\max_{q,w,\theta_0} \int_{\theta_0}^{\bar{\theta}} \sigma(\theta)[(1-\rho)q(\theta) + \rho P] \, \mathrm{d}F(\theta) \tag{30}$$

subject to (MPS') and (BP') on $\theta \in [\theta_0, \bar{\theta}]$, and

$$U(\theta) \equiv [w(\theta) - c(q(\theta),\theta) - P]\sigma(\theta) + w_\varnothing(1-\sigma(\theta)) = -\int_{\underline{\theta}}^{\theta} \sigma(x) c_\theta(q(x),x) \, \mathrm{d}x \quad \text{(IC-Env)}$$

$q(\theta)$ increasing (IC-Mon)

$$U(\theta) = [w(\theta) - c(q(\theta),\theta) - P]\sigma(\theta) + w_\varnothing(1-\sigma(\theta)) \geq 0 \quad \text{(IR)}$$

The following proposition characterizes the optimal rating scheme when the principal cares sufficiently about the certification fee.

Proposition 11. *Assume* ρ *is sufficiently close to 1 or* $c(q,\theta)$ *is sufficiently convex in* q. *Then,* *the optimal rating induces the quality scheme $q^*(\theta)$ given by*

$$\begin{cases} c_q(q^*(\theta),\theta) = \frac{1-\rho}{\rho} + \frac{1-F(\theta)}{f(\theta)} c_{\theta q}(q^*(\theta),\theta) \text{ (“}\geq\text{” if } c(q^*(\theta),\theta)=0\text{)}, & \text{if } \theta \geq \theta_0^*, \\ c(q^*(\theta),\theta) = 0, & \text{otherwise}, \end{cases}$$

where the optimal cutoff is

$$\theta_0^* = \sup\{\theta \in [\underline{\theta}, \bar{\theta}] : (1-\rho)q^*(\theta) + (\rho - 1)\theta - \rho c(q^*(\theta),\theta) + \rho \frac{1-F(\theta)}{f(\theta)} c_\theta(q^*(\theta),\theta) \leq 0\}.$$

The optimal certification fee is

$$P^* = \mathbf{E}[\theta] - \mathbf{E}\left[c(q^*(\theta),\theta) - \frac{1-F(\theta)}{f(\theta)} c_\theta(q^*(\theta),\theta) \mid \theta \geq \theta_0^*\right],$$

and the optimal rating is a noisy test characterized by the interim wage

$$w^*(\theta) = \begin{cases} \int_{\theta_0^*}^{\theta} c_q(q^*(\theta),\theta) q^{*\prime}(\theta)\, \mathrm{d}\theta + P^*, & \text{if } \theta \in [\theta_0^*, \bar{\theta}], \\ \mathbf{E}[\theta \mid \theta \leq \theta_0^*], & \text{otherwise}, \end{cases}$$

which satisfies $w^{*\prime}(\theta) = c_q(q^*(\theta),\theta)q^{*\prime}(\theta) < 1$ when ρ is sufficiently close to 1 or $c(q,\theta)$ is sufficiently convex in q.

Corollary 11.1. *When $\rho = 1$, that is, the principal (e.g., a monopoly certifier) cares only about the certification fee, the optimal rating is to give the same score to every participant, thereby inducing no effort ($c(q^*(\theta),\theta) = 0$) and full participation ($\sigma^*(\theta) = 1$ for all $\theta \in [\underline{\theta}, \bar{\theta}]$). The optimal fee is $P^* = \mathbf{E}[\theta]$.*

Remark 9. The monopoly certifier extracts the entire surplus by setting a certification fee $P^* = \mathbf{E}[\theta]$, which is reminiscent of the result of Lizzeri (1999), where there is no ability signaling through effort. There also exist other equilibria where agents invest no effort and the cutoff is $\theta_0^* \in [\underline{\theta}, \bar{\theta}]$—in particular, an equilibrium where no agents take the test (Cf. Lizzeri, 1999; Ali, Haghpanah, Lin and Siegel, 2022).

7 Conclusion

Ratings are often used to motivate agent performance or the firm investment in product quality, particularly when monetary transfers are limited. When the market rewards agents based on the perception of their endogenous quality or exogenous abilities, ratings can provide reputational incentives in place of monetary incentives. In this paper, I study the optimal rating scheme to incentivize agent investment in quality, when they have private information about their costs of investment. I provide necessary and sufficient conditions under which pass fail tests and lower censorship are optimal within deterministic ratings. Beyond lower censorship, I also solve for the optimal deterministic ratings for general preferences and distributions. In particular, when the principal's objective is expected quality, lower censorship is optimal if and only if types are concentrated around the mode of the distribution (i.e., density is quasi-unimodal), and pass/fail tests are optimal if types are concentrated towards the top (i.e., density is quasi-increasing).

The deterministic rating design problem is equivalent to a delegation problem with voluntary participation (Amador and Bagwell, 2022). My results improve upon the existing results by providing weaker sufficient conditions for lower censorship (corresponding to threshold delegation) that are also necessary in the linear delegation case. I also provide necessary and sufficient conditions for pass/fail tests (corresponding to take-it-or-leave-it offers in delegation) to be optimal. The results allows for general state-dependent preferences of the principal and nonlinear delegation. Additionally, through the equivalence established by Kolotilin and Zapechelnyuk (2019), the results also have implications for the Bayesian persuasion literature, especially in cases where the sender's payoffs are nonlinear in the state.

By defining an interim wage function and characterizing the necessary and sufficient condition for an incentive-compatible direct mechanism to be feasible (i.e., can be induced by a rating scheme), I use an interim approach to the rating design problem. The interim approach is particularly useful in solving for the optimal general (possibly stochastic) rating schemes, as it reduces the rating design problem to the optimization over interim wage functions rather than ratings themselves.

When stochastic rating schemes are allowed, I also provide sufficient conditions under which pass/fail tests remain optimal. In the quality maximization case, a pass/fail test is optimal if the ability density is increasing. Moreover, I identify conditions under which stochastic ratings can strictly improve on deterministic ratings. For example, a noisy test that partially pools low quality with high quality enables the principal to increase the incentives for low types at the cost of incentives for high types, which can increase the overall expected quality if the ability density has a heavier tail than Pareto distribution—in other words, they are a few very high ability agents.

Nevertheless, while I provide sufficient conditions for pass/fail tests and lower censorship to remain optimal, I have not characterized the optimal ratings in general when stochastic ratings are feasible. Further, in the current model, the market either values the agent's endogenous quality (i.e., productive investment) or exogenous abilities (i.e., ability signaling), but a combination of both cases is not considered. One would expect a combination of them makes the full revelation of quality more likely to be optimal than pure productive investment and less likely than pure ability signaling. Moreover, while I focus on the case where agents can choose quality deterministically, a more general case where investing effort increases quality stochastically is also worth exploring (e.g., Saeedi and Shourideh, 2023), where the moral hazard problem becomes more complicated. In addition, competition among certifiers (i.e., test designers) is also a direction for future research.

Appendix A Setup of the Hamiltonian

A.1 Deterministic Ratings (Section 4)

Define $U(\theta) = \int_{\underline{\theta}}^{\theta} q(x)dx + \underline{U}$. Rewrite the constraints in [P'] as

$$\theta q(\theta) - c(q(\theta)) = U(\theta) \tag{A.1}$$

$$\dot{U} = q(\theta) \tag{A.2}$$

$$\dot{q} = \nu(\theta) \geq 0 \tag{A.3}$$

$$q \text{ increasing} \tag{A.4}$$

$$U(\underline{\theta}), q(\underline{\theta}) \geq 0 \tag{A.5}$$

$$U(\bar{\theta}), q(\bar{\theta}) \text{ free.} \tag{A.6}$$

Set up the Hamiltonian

$$H = v(q(\theta), \theta) f(\theta) + \gamma(\theta)[\theta q(\theta) - c(q(\theta)) - U(\theta)] + \Gamma(\theta) q(\theta) + \mu(\theta)\nu(\theta) \tag{A.7}$$

where U, q are state variables and ν is the control variable; Γ is Hamiltonian multiplier on $\dot{U}$ and μ is Hamiltonian multiplier on $\dot{q}$; γ is the Lagrangian multiplier on $U = \theta q - c(q)$.[77]

By the Pontryagin's maximum principle (Hellwig, 2010, Theorem 4.1),

$$-\frac{\partial H}{\partial q} = -(v_q f + \gamma(\theta - c'(q)) + \Gamma) = \dot{\mu} \tag{A.8}$$

$$-\frac{\partial H}{\partial U} = \gamma = \dot{\Gamma} \tag{A.9}$$

$$\frac{\partial H}{\partial \nu} = \mu \leq 0, \quad \mu(\theta) = 0 \text{ if } q \text{ is strictly increasing at } \theta,\text{[78]} \tag{A.10}$$

$$\Gamma(\underline{\theta}) \leq 0, \quad \Gamma(\underline{\theta}) U(\underline{\theta}) = 0 \tag{A.11}$$

$$\mu(\underline{\theta}) \leq 0, \quad \mu(\underline{\theta}) q(\underline{\theta}) = 0 \tag{A.12}$$

$$\Gamma(\bar{\theta}) = 0, \quad \mu(\bar{\theta}) = 0. \tag{A.13}$$

Because $c'(q_f(\theta)) = \theta$ (and thus $\dot{q}_f(\theta) > 0$), on the fully revealing region where $q(\theta) = q_f(\theta)$, we have $\Gamma(\theta) = -v_q(q_f(\theta), \theta) f(\theta)$.

At θ_0, the switching condition (Bryson and Ho, 1975, Chapter 3.7) (see also Clarke,

[77] Note $U = \theta q - c(q)$ is a pure state constraint (i.e., containing no control variable). Therefore, the multipliers Γ and μ can be discontinuous at junction points between intervals on which the pure state constraint is binding and intervals on which it is not (Seierstad and Sydsaeter, 1977).

[78] An increasing function q is said to be *strictly* increasing at θ if $q(\theta + \varepsilon) - q(\theta + \varepsilon) > 0$ for all $\varepsilon > 0$.

2013, Chapter 22.5 for the hybrid maximum principle)

$$\Gamma(\theta_0+) = \Gamma(\theta_0-) \tag{A.14}$$

$$H(\theta_0+) = v(q_i(\theta_0), \theta_0) f(\theta_0) + \Gamma(\theta_0+) q_i(\theta_0) = H(\theta_0-) = 0, \tag{A.15}$$

By Kamien and Schwartz (1971, 2012), sufficiency requires the maximized Hamiltonian $\bar{H}(q, U, \gamma, \mu, \Gamma) = \max_\nu H(q, U, \nu, \gamma, \mu, \Gamma)$ to be concave in (q, U) for given (γ, μ, Γ), which requires $v_{qq} f - \gamma c''(q) \leq 0$. Define $\kappa = \inf_{q,\theta}\{-v_{qq}/c''(q)\}$. Concavity is satisfied if $\Gamma + \kappa F$ is increasing.

The proposed multipliers for the Hamiltonian equation (A.7), according to the Pontryagin's maximum principle, are

$$\Gamma(\theta) = \begin{cases} -A - \kappa(F(\theta) - F(\theta_0)), & \text{if } \theta \in [\underline{\theta}, \theta_L(\theta_0)] \\ -v_q(\theta, q_f(\theta)) f(\theta), & \text{if } \theta \in (\theta_L(\theta_0), \bar{\theta}) \\ 0, & \text{if } \theta = \bar{\theta} \end{cases} \tag{A.16}$$

$$\gamma(\theta) = \begin{cases} -\kappa f(\theta), & \text{if } \theta \in [\underline{\theta}, \theta_L(\theta_0)] \\ -[v_q(\theta, q_f(\theta)) f(\theta)]', & \text{if } \theta \in (\theta_L(\theta_0), \bar{\theta}] \end{cases} \tag{A.17}$$

and

$$\mu(\theta) = \begin{cases} \int_\theta^{\theta_0} v_q(0, \tilde{\theta}) f(\tilde{\theta})\, \mathrm{d}\tilde{\theta} - \kappa\theta(F(\theta_0) - F(\theta)) - (\theta_0 - \theta)A \leq 0, & \text{if } \theta \in [\underline{\theta}, \theta_0] \\ -\int_{\theta_0}^{\theta} v_q(q_i(\theta_0), \tilde{\theta}) f(\tilde{\theta})\, \mathrm{d}\tilde{\theta} + \kappa(\theta - \theta_c(\theta_0))(F(\theta) - F(\theta_0)) + (\theta - \theta_0)A \leq 0, & \text{if } \theta \in (\theta_0, \theta_L(\theta_0)] \\ 0, & \text{if } \theta \in (\theta_L(\theta_0), \bar{\theta}] \end{cases} \tag{A.18}$$

where $\theta_c(\theta_0) \equiv c'(q_i(\theta_0))$ and the multiplier

$$A = \frac{1}{\theta_c(\theta_0) - \theta_0} \int_{\theta_0}^{\theta_L(\theta_0)} v_q(q_i(\theta_0), \theta)\, \mathrm{d}F(\theta). \tag{A.19}$$

See Appendix C for the comparison with the multipliers proposed by Amador and Bagwell (2022).

A.2 General Ratings (Section 5)

Define $D(\theta) = \int_{\underline{\theta}}^{\theta} (w(\theta') - q(\theta'))\,\mathrm{d}F(\theta') \geq 0$ and $U(\theta) = \int_{\underline{\theta}}^{\theta} w(x)dx + \underline{U}$. Rewrite the constraints as

$$D(\theta) \geq 0 \text{ (MPS)} \tag{A.20}$$

$$\dot{D} = [w(\theta) - q(\theta)]f(\theta) \tag{A.21}$$

$$\theta w(\theta) - c(q(\theta)) = U(\theta) \tag{A.22}$$

$$\dot{U} = w(\theta) \tag{A.23}$$

$$\dot{q} = \nu \geq 0 \tag{A.24}$$

$$U(\underline{\theta}), q(\underline{\theta}) \geq 0,\ \ D(\underline{\theta}) = 0 \tag{A.25}$$

$$U(\bar{\theta}), q(\bar{\theta}) \text{ free},\ \ D(\bar{\theta}) = 0 \text{ (BP)} \tag{A.26}$$

Set up the Hamiltonian

$$\begin{aligned} H = v(q(\theta),\theta)f(\theta) + \gamma(\theta)[\theta w(\theta) - c(q(\theta)) - U(\theta)] + \lambda(\theta)D(\theta) \\ +\Lambda(\theta)[w(\theta) - q(\theta)]f(\theta) + \Gamma(\theta)w(\theta) + \mu(\theta)\nu(\theta) \end{aligned} \tag{A.27}$$

where U, q, D are state variables and w, ν is the control variable; $\lambda(\theta)$ is the Lagrangian multiplier on $D(\theta) \geq 0$ (MPS), $\gamma(\theta)$ is the Lagrangian multiplier on $U(\theta) = \theta w(\theta) - c(q(\theta))$, Λ is the Hamiltonian multiplier on $\dot{D} = [w(\theta) - q(\theta)]f(\theta)$, and Γ is the Hamiltonian multiplier on $\dot{U} = w(\theta)$.[79]

By the Pontryagin's maximum principle,

$$-\frac{\partial H}{\partial q} = -(v_q f - \gamma c'(q) - \Lambda f) = \dot{\mu} \tag{A.28}$$

$$-\frac{\partial H}{\partial D} = -\lambda = \dot{\Lambda} \tag{A.29}$$

$$-\frac{\partial H}{\partial U} = \gamma = \dot{\Gamma} \tag{A.30}$$

$$\frac{\partial H}{\partial w} = \theta\gamma + \Lambda f + \Gamma = 0 \tag{A.31}$$

$$\frac{\partial H}{\partial \nu} = \mu \leq 0, \quad \mu(\theta) = 0 \text{ if } q \text{ is strictly increasing at } \theta \tag{A.32}$$

$$\lambda(\theta) \geq 0, \quad \lambda(\theta)D(\theta) = 0 \tag{A.33}$$

$$\Gamma(\underline{\theta}) \leq 0, \quad \Gamma(\underline{\theta})U(\underline{\theta}) = 0 \tag{A.34}$$

[79] Now $D \geq 0$ is a pure state constraint, and $U = \theta w - c(q)$ is not. Therefore, the multipliers Λ can be discontinuous at junction points between intervals on which $D \geq 0$ is binding and intervals on which it is not (Seierstad and Sydsaeter, 1977).

$$\mu(\underline{\theta}) \le 0, \quad \mu(\underline{\theta})q(\underline{\theta}) = 0 \tag{A.35}$$

$$\Gamma(\bar{\theta}) = 0, \quad \mu(\bar{\theta}) = 0 \tag{A.36}$$

$$\Lambda(\bar{\theta}) \text{ no condition.} \tag{A.37}$$

The conditions imply

$$[\theta\Gamma(\theta)]' = \theta\gamma + \Gamma = -\Lambda(\theta)f(\theta) \tag{A.38}$$

$$\dot{\mu} = -[v_q(q(\theta),\theta)f(\theta) + \gamma(\theta)(\theta - c'(q)) + \Gamma(\theta)] \tag{A.39}$$

$$\dot{\Lambda}(\theta) = -\lambda(\theta) \le 0, \quad \lambda(\theta)\int_{\underline{\theta}}^{\theta} (w(\theta') - q(\theta'))\,\mathrm{d}F(\theta') = 0 \tag{A.40}$$

On the fully revealing region where $q(\theta) = q_f(\theta)$, we have $\Gamma(\theta) = -v_q(q_f(\theta),\theta)f(\theta)$, as in the deterministic case.

By Kamien and Schwartz (1971), sufficiency requires the maximized Hamiltonian $\bar{H}(q,U,D,\gamma,\mu,\Gamma,\lambda,\Lambda) = \max_{\nu,w} H(q,U,D,\nu,w,\gamma,\mu,\Gamma,\lambda,\Lambda)$ to be concave in (q,U,D) for given $(\gamma,\mu,\Gamma,\lambda,\Lambda)$, which requires $v_{qq}f - \gamma c''(q) \le 0$. Recall that $\kappa = \inf_{q,\theta}\{-v_{qq}/c''(q)\}$. Concavity is satisfied if $\Gamma + \kappa F$ is increasing.

A.3 Ability Signaling (Section 6)

The setup of Hamiltonian is almost identical to Appendix A.2, except that the state equation of D is replaced by $\dot{D} = [w(\theta) - \theta]f(\theta)$ due to (MPS').[80] Set up the Hamiltonian

$$\begin{aligned} H = v(q(\theta),\theta)f(\theta) + \gamma(\theta)[\theta w(\theta) - c(q(\theta)) - U(\theta)] + \lambda(\theta)D(\theta) \\ +\Lambda(\theta)[w(\theta) - \theta]f(\theta) + \Gamma(\theta)w(\theta) + \mu(\theta)\nu(\theta) \end{aligned} \tag{A.41}$$

By the Pontryagin's maximum principle,

$$-\frac{\partial H}{\partial q} = -(v_q f - \gamma c'(q)) = \dot{\mu} \tag{A.42}$$

$$-\frac{\partial H}{\partial D} = -\lambda = \dot{\Lambda} \tag{A.43}$$

$$-\frac{\partial H}{\partial U} = \gamma = \dot{\Gamma} \tag{A.44}$$

$$\frac{\partial H}{\partial w} = \theta\gamma + \Lambda f + \Gamma = 0 \tag{A.45}$$

$$\frac{\partial H}{\partial \nu} = \mu \le 0, \quad \mu(\theta) = 0 \text{ if } q \text{ is strictly increasing at } \theta \tag{A.46}$$

[80]Consequently, $D \ge 0$ is no longer a pure state constraint.

$$\lambda(\theta) \geq 0, \quad \lambda(\theta)D(\theta) = 0 \tag{A.47}$$

$$\Gamma(\underline{\theta}) \leq 0, \quad \Gamma(\underline{\theta})U(\underline{\theta}) = 0 \tag{A.48}$$

$$\mu(\underline{\theta}) \leq 0, \quad \mu(\underline{\theta})q(\underline{\theta}) = 0 \tag{A.49}$$

$$\Gamma(\bar{\theta}) = 0, \quad \mu(\bar{\theta}) = 0 \tag{A.50}$$

$$\Lambda(\bar{\theta}) \text{ no condition.} \tag{A.51}$$

On the fully revealing region where $q(\theta) = q_f(\theta)$, we have $\gamma(\theta) = v_q(q_f(\theta), \theta)f(\theta)/c'(q_f(\theta))$ and $\Gamma(\theta) = -\int_\theta^{\bar{\theta}} v_q(q_f(x), x)f(x)/c'(q_f(x))dx$, and thus

$$-\Lambda(\theta) = -\frac{\theta\gamma + \Gamma}{f} = \frac{v_q(q_f(\theta), \theta)\theta}{c'(q_f(\theta))} - \frac{\int_\theta^{\bar{\theta}} v_q(q_f(x), x)f(x)/c'(q_f(x))dx}{f(\theta)} \equiv J(\theta|q_f) \tag{A.52}$$

When $J(\theta|q_f) = -\Lambda(\theta)$ is increasing in θ, $\lambda(\theta) = -\Lambda'(\theta) \geq 0$.

By Kamien and Schwartz (1971), sufficiency requires $v_{qq}f - \gamma c''(q) \leq 0$. Substituting $\gamma(\theta) = v_q(q_f(\theta), \theta)f(\theta)/c'(q_f(\theta))$ into it, we have

$$v_{qq}(q_f(\theta), \theta) - v_q(q_f(\theta), \theta) \cdot c''(q_f(\theta))/c'(q_f(\theta)) \leq 0, \tag{A.53}$$

which always holds because $v_q(q_f(\theta), \theta) \geq 0$ and $v_{qq} \leq 0$.

Appendix B Proofs

B.1 Proofs of Section 3

B.1.1 Incentive Compatibility

Lemma B.1. *A direct mechanism $(q(\theta), w(\theta))$ is incentive compatible if and only if*

- *$q(\theta)$ is increasing, and*
- *$U(\theta) \equiv w(\theta) - c(q(\theta), \theta) = -\int_{\underline{\theta}}^{\theta} c_\theta(q(x), x)dx + \underline{U}$,*

where $\underline{U} = w(\underline{\theta}) - c(q(\underline{\theta}), \underline{\theta})$. These conditions also imply $w(\theta)$ is increasing.

Proof. (IC) is $\theta \in \arg\max_{\hat{\theta}}\{w(\hat{\theta}) - c(q(\hat{\theta}), \theta)\}$. Proof is standard by noting that $U(\theta) = \max_{\hat{\theta}}\{w(\hat{\theta}) - c(q(\hat{\theta}), \theta)\}$. □

B.1.2 Proof of Corollary 1.1

Proof of Corollary 1.1. Because $w(\theta) - q(\theta)$ is decreasing, there exists a point $\hat{\theta} \in [\underline{\theta}, \bar{\theta}]$ such that $w(\theta) \leq q(\theta)$ for all $\theta > \hat{\theta}$ and $w(\theta) \geq q(\theta)$ for all $\theta < \hat{\theta}$ (i.e., w single-crosses q from above). Thus, $D(\theta) \equiv \int_{\underline{\theta}}^{\theta} (w(\theta) - q(\theta))\, \mathrm{d}F(\theta) \geq 0$ for all $\theta < \hat{\theta}$, and $D'(\theta) = (w(\theta) - q(\theta)) f(\theta) \leq 0$ for all $\theta > \hat{\theta}$. Because $D(\bar{\theta}) = 0$ (BP), we have $D(\theta) \geq 0$ (MPS). □

B.2 Proofs of Section 4

B.2.1 Proof of Lemmas 3–6

Proof of Lemma 3. Under a deterministic rating scheme π, if the rating maps a (potentially singleton) nonempty set of quality to the same score s, only $\hat{q}(s) \equiv \min\{q : \pi(q) = s\}$ will be chosen by an agent. Thus, for any $q \in \{\hat{q}(s) : s \in \pi(Q)\}$ (where $\pi(Q) \equiv \{\pi(q) : q \in Q\}$) chosen by an agent, the interim wage is $\hat{w}(q) = \mathbf{E}[\tilde{q} \mid s = \pi(q)] = q$. Therefore, for any $\theta \in [\underline{\theta}, \bar{\theta}]$, the interim wage is $w(\theta) \equiv \hat{w}(q(\theta)) = q(\theta)$. □

Proof of Lemma 4. Because $q(\theta)$ is increasing, it has at most countably many jump discontinuities and is differentiable almost everywhere. Assume without loss that $q(\theta)$ is right-continuous, so that the right-derivative $q'(\theta+) \equiv \lim_{h \to 0^+} \frac{q(\theta+h)-q(\theta)}{h}$ always exists. Then, $\theta \in \arg\max_{\hat{\theta}} \{w(\hat{\theta}) - c(q(\hat{\theta}))/\theta\}$ (IC) implies $(c'(q(\theta)) - \theta) q'(\theta) = 0$, so either $q(\theta) = q_f(\theta)$ or $q'(\theta) = 0$.

At each discontinuity, condition 1 and 3 follow from the continuity of $U(\theta)$ (because $U(\theta) = \max_{\hat{\theta}} \{q(\hat{\theta}) - c(q(\hat{\theta}), \theta)\}$ is convex, it is absolutely continuous). Condition 2 follows from the first part ($q'(\theta) = 0$) and continuity of $U(\theta)$ (which determines the interval endpoints). □

Proof of Lemma 5. By (IR) and (IC), there exists a cutoff type $\theta_0 \in [\underline{\theta}, \bar{\theta}]$ such that $U(\theta) \geq 0$ if and only if $\theta \geq \theta_0$. If $\theta < \theta_0$, then $U(\theta) < 0$, so the agent chooses $q(\theta) = 0$. If $\theta > \theta_0$, then $U(\theta) < 0$ and thus $q(\theta) > 0$. If $\theta_0 \in (\underline{\theta}, \bar{\theta})$ is in the interior, then $U(\theta_0) = 0$, so the agent is indifferent between $q_i(\theta_0)$ and $q = 0$. □

Proof sketch of Lemma 6. If $r(\theta)$ is unimodal with mode $\theta_m \in (\underline{\theta}, \bar{\theta})$, then $R(\theta)$ is convex-concave on $[\underline{\theta}, \bar{\theta}]$ with a reflection point θ_m. Note that $r(\theta) = -\kappa(1 - F(\theta_0)) \leq 0$ for all $\theta \geq \bar{\theta}$, so R is decreasing for all $\theta \geq \bar{\theta}$. Therefore, it satisfies conditions (S) and (C) at some $\theta_0 \in (0, \theta_m)$ such that $\theta_c(\theta_0) \geq \theta_m$.

An increasing $r(\theta)$ satisfies condition (S) at some θ_0 such that $\theta_c(\theta_0) \geq \bar{\theta}$ and thus satisfies condition (C) vacuously. Note that $r(\theta) = -\kappa(1 - F(\theta_0)) \leq 0$ for all $\theta \geq \bar{\theta}$, so R is decreasing for all $\theta \geq \bar{\theta}$. For any increasing $r(\theta)$, $\phi(\theta_0) \equiv R(\theta_c(\theta_0)) - R(\theta_0) - (\theta_c(\theta_0) - \theta_0) r(\theta_0)$

is strictly decreasing on $\theta \in [\theta_c^{-1}(\bar{\theta}), \bar{\theta}]$ because $\phi'(\theta_0) = \theta_c'(\theta_0) r(\theta_c(\theta_0)) - \theta_c'(\theta_0) r(\theta_0) - r'(\theta_0)(\theta_c(\theta_0) - \theta_0) < 0$. Also, $\phi(\bar{\theta}) < 0$ and $\phi(\theta_c^{-1}(\bar{\theta})) \geq 0$, so $\phi(\theta_0) = 0$ has a unique solution $\theta_0^* \in [\theta_c^{-1}(\bar{\theta}), \bar{\theta})$ such that $\theta_c(\theta_0^*) \geq \bar{\theta}$. For $\theta < \theta_c^{-1}(\bar{\theta})$, there cannot be a θ_0 that satisfies (C) because $r(\theta)$ is increasing on $[\theta_c(\theta_0), \bar{\theta}]$.

A decreasing $r(\theta)$ satisfies conditions (S) and (C) at $\theta_0 = \underline{\theta}$. If $\underline{\theta} = 0$, then $\theta_c(\underline{\theta}) = 0$, so a quasi-decreasing function is decreasing by condition (C). □

B.2.2 Proof of Proposition 3

Definition of $r(\theta)$ under Condition (LD). To make the notation consistent with the Hamiltonian in Appendix A.1, I rewrite $r(\theta)$ in the general form. Recall that the "relative concavity" of the principal and agent's preferences is

$$\kappa = \inf_{q \in Q, \theta \in [\underline{\theta}, \bar{\theta}]} \{-v_{qq}(q, \theta)/c''(q)\} = \alpha, \tag{B.1}$$

Define

$$\begin{aligned} r(\theta) &= (\beta(\theta) - \alpha\theta) f(\theta) - \alpha(F(\theta) - F(\theta_0)) \\ &= \begin{cases} v_q(0, \theta) f(\theta) - \kappa\theta f(\theta) - \kappa(F(\theta) - F(\theta_0)) \\ v_q(q_i(\theta_0), \theta) f(\theta) - \kappa(\theta - \theta_c(\theta_0)) f(\theta) - \kappa(F(\theta) - F(\theta_0)) \\ v_q(q_f(\theta), \theta) f(\theta) - \kappa(F(\theta) - F(\theta_0)). \end{cases} \end{aligned}$$

Define

$$\begin{aligned} L(\theta|\theta_0) &= \frac{1}{\theta_0 - \theta} \int_\theta^{\theta_0} r(\tilde{\theta}) \,\mathrm{d}\tilde{\theta} \\ &= \begin{cases} \frac{1}{\theta_0 - \theta} \left[\int_\theta^{\theta_0} v_q(0, \tilde{\theta}) f(\tilde{\theta}) \,\mathrm{d}\tilde{\theta} - \kappa\theta(F(\theta_0) - F(\theta)) \right], & \text{if } \theta \in [\underline{\theta}, \theta_0), \\ \frac{1}{\theta - \theta_0} \left[\int_{\theta_0}^{\theta} v_q(q_i(\theta_0), \tilde{\theta}) f(\tilde{\theta}) \,\mathrm{d}\tilde{\theta} - \kappa(\theta - \theta_c(\theta_0))(F(\theta) - F(\theta_0)) \right], & \text{if } \theta \in (\theta_0, \theta_c(\theta_0)]. \end{cases} \end{aligned} \tag{B.2}$$

Define the multiplier $A = L(\theta_c(\theta_0)|\theta_0)$.

Recall that under Condition (LD), $v(q, \theta) = \beta(\theta) q - \alpha c(q)$, so $v_{qq}(q, \theta) + \kappa c''(q) = 0$.

Proof of Proposition 3. (Sufficiency). First, I show that the point θ_0 at which conditions (S) and (C) hold coincide with the optimal cutoff that satisfies equation (OPT).

Lemma B.2. *Conditions* (S) *and* (C) *hold at some* θ_0 *if and only if* θ_0 *satisfies equation* (OPT).

Proof of Lemma B.2. Recall that $r(\theta) = (\beta(\theta) - \alpha\theta)f(\theta) - \alpha(F(\theta) - F(\theta_0))$. Thus,

$$v(q_i(\theta), \theta) = \beta(\theta)q_i(\theta) - \alpha c(q_i(\theta)) = [r(\theta) + \alpha(F(\theta) - F(\theta_0))]\,\frac{q_i(\theta)}{f(\theta)}$$

because $c(q_i(\theta)) = \theta q_i(\theta)$.

If condition (S) holds at some $\theta_0 > 0$, then $A = r(\theta_0)$, so $V'(\theta_0) = A \cdot q_i(\theta_0) - v(q_i(\theta_0), \theta_0)f(\theta_0) = 0$ (OPT). If conditions (S) and (C) hold at $\theta_0 = 0$, then $V'(0) = A(0) \cdot q_i(0) - v(q_i(0), 0)f(0) \leq 0$ because $q_i(0) = 0$. □

Then, I use the optimal control method, where the proposed multipliers are in Section A.1. The following necessary conditions need to be verified:

- Condition (S1) guarantees that $\mu(\theta) \leq 0$ on $(\theta_0^*, \theta_L(\theta_0^*)]$ (in equation (A.18)), so that $q^*(\theta) = q_i(\theta_0^*)$ is constant on $(\theta_0^*, \theta_L(\theta_0^*)]$. Recall that

$$\mu(\theta) = \begin{cases} \int_\theta^{\theta_0} v_q(0, \tilde{\theta})f(\tilde{\theta})\,\mathrm{d}\tilde{\theta} - \kappa\theta(F(\theta_0) - F(\theta)) - (\theta_0 - \theta)A \leq 0, & \text{if } \theta \in [\underline{\theta}, \theta_0] \\ -\int_{\theta_0}^{\theta} v_q(q_i(\theta_0), \tilde{\theta})f(\tilde{\theta})\,\mathrm{d}\tilde{\theta} + \kappa(\theta - \theta_c(\theta_0))(F(\theta) - F(\theta_0)) + (\theta - \theta_0)A \leq 0, & \text{if } \theta \in (\theta_0, \theta_L(\theta_0)) \\ 0, & \text{if } \theta \in (\theta_L(\theta_0), \bar{\theta}] \end{cases} \tag{B.3}$$

- Condition (S2) guarantees that $\mu(\theta) \leq 0$ on $(\underline{\theta}, \theta_0^*]$, so that $q^*(\theta) = 0$ is constant on $(\underline{\theta}, \theta_0^*]$.

- Condition (S1) implies that $\mu(\theta_L(\theta_0^*)) = 0$ (so μ is continuous at $\theta_L(\theta_0^*)$), and thus it is compatible with $q(\theta)$ being strictly increasing at $\theta_L(\theta_0^*)$ (as $q(\theta) = q_f(\theta)$ on $[\theta_L(\theta_0^*), \bar{\theta}]$).

Furthermore, sufficient condition (concavity) requires that $\Gamma + \kappa F$ is increasing.

- Condition (C) implies $\Gamma + \kappa F$ is increasing on $(\theta_L(\theta_0^*), \bar{\theta}]$ where $q^*(\theta) = q_f(\theta)$.

- The jumps of Γ are nonnegative at $\theta_L(\theta_0^*)$ by condition (S1) if $\theta_L(\theta_0^*) < \bar{\theta}$ (by $A \geq 0$ if $\theta_L(\theta_0^*) = \bar{\theta}$) and nonnegative at $\bar{\theta}$ by Assumption 1.

By Lemma B.2, θ_0^* is the optimal cutoff that satisfies the switching condition

$$H(\theta_0+) = v(q_i(\theta_0), \theta_0)f(\theta_0) + \Gamma(\theta_0+)q_i(\theta_0) = H(\theta_0-) = 0,$$

where $\Gamma(\theta_0+) = -A$ as in equation (A.16).

(Necessity). Necessity can be shown à la Amador and Bagwell (2013, Proposition 2) using perturbation arguments. □

B.2.3 Proofs of Observations 5–7

Proof of Observation 5. By definition,

$$\begin{aligned} A(\theta_0) &= \frac{1}{\theta_c(\theta_0)-\theta_0}\int_{\theta_0}^{\theta_c(\theta_0)} r(\tilde{\theta})\,\mathrm{d}\tilde{\theta} = \frac{1}{\theta_c(\theta_0)-\theta_0}\int_{\theta_0}^{\theta_c(\theta_0)} v_q(q_i(\theta_0),\theta)\,\mathrm{d}F(\theta) \\ &= \frac{1}{\theta_c(\theta_0)-\theta_0}\int_{\theta_0}^{\theta_L(\theta_0)} v_q(q_i(\theta_0),\theta)\,\mathrm{d}F(\theta). \end{aligned}$$

The second equality follows from integration by parts (see equation (18) for the general case):

$$\begin{aligned} \int_{\theta_0}^{\theta} r(\tilde{\theta})\,\mathrm{d}\tilde{\theta} &= \int_{\theta_0}^{\theta}[(\beta(\tilde{\theta})-\alpha\theta)f(\tilde{\theta})-\alpha(F(\tilde{\theta})-F(\theta_0))]\,\mathrm{d}\tilde{\theta} \\ &= \int_{\theta_0}^{\theta} v_q(q_i(\theta_0),\tilde{\theta})\,\mathrm{d}F(\tilde{\theta})-\kappa(\theta-\theta_c(\theta_0))(F(\theta)-F(\theta_0)). \end{aligned} \tag{B.4}$$

By convention, the CDF $F(\theta)=1$ for all $\theta \geq \bar{\theta}$ and $F(\theta)=0$ for all $\theta \leq \underline{\theta}$. Therefore, the upper bound of the integration can be changed from $\theta_c(\theta_0)$ to $\theta_L(\theta_0)=\max\{\min\{\theta_c(\theta_0),\bar{\theta}\},\underline{\theta}\}$ because $F(\theta)=1$ on $(\theta_L(\theta_0),\theta_c(\theta_0)]$ if $\theta_c(\theta_0)>\theta_L(\theta_0)$. □

Proof of Observation 6. If $v_q(q_i(\theta),\theta)\geq 0$ (and $\not\equiv 0$) for all $\theta\in[\underline{\theta},\theta_c(\underline{\theta})]$, then $v_q(q_i(\underline{\theta}),\theta)\geq 0$ for all $\theta\in[\underline{\theta},\theta_c(\underline{\theta})]$, so $A(\theta)>0$ for all $\theta\leq\underline{\theta}$. Thus, for all $\theta_0<\underline{\theta}$, $f(\theta_0)=0$ implies $V'(\theta_0)=A(\theta_0)q_i(\theta_0)-v(q_i(\theta_0),\theta_0)f(\theta_0)>0$. Hence, $\theta_0^*\theta 0$. □

Proof of Observation 7. Recall that $\kappa=\inf_{q,\theta}\{-v_{qq}/c''(q)\}$. If $v_{q\theta}(q,\theta)\leq\kappa$, then $d(q,\theta)=v(q,\theta)-\kappa(\theta q-c(q))$ satisfies $d_{qq}\leq 0$ and $d_{q\theta}\leq 0$. Therefore,

$$\begin{aligned} \int_{\theta_0}^{\theta_L(\theta_0)} v_q(q_i(\theta_0),\theta)\,\mathrm{d}\theta &= \int_{\theta_0}^{\theta_L(\theta_0)} d_q(q_i(\theta_0),\theta)\,\mathrm{d}\theta+\int_{\theta_0}^{\theta_L(\theta_0)}\kappa(\theta-c'(q))\,\mathrm{d}\theta \\ &\leq \int_{\theta_0}^{\theta_L(\theta_0)} d_q(q_i(\theta_0),\theta)\,\mathrm{d}\theta \leq d_q(q_i(\theta_0),\theta_0)(\theta_L(\theta_0)-\theta_0) \\ &\leq \frac{d(q_i(\theta_0),\theta_0)}{q_i(\theta_0)}(\theta_L(\theta_0)-\theta_0)=\frac{v(q_i(\theta_0),\theta_0)}{q_i(\theta_0)}(\theta_L(\theta_0)-\theta_0) \end{aligned} \tag{B.5}$$

Then, because f is decreasing,

$$\begin{aligned} \int_{\theta_0}^{\theta_L(\theta_0)} v_q(q_i(\theta_0),\theta)f(\theta)\,\mathrm{d}\theta &\leq f(\theta_0)\int_{\theta_0}^{\theta_L(\theta_0)} v_q(q_i(\theta_0),\theta)\,\mathrm{d}\theta \\ &\leq \frac{v(q_i(\theta_0),\theta_0)}{q_i(\theta_0)}f(\theta_0)(\theta_L(\theta_0)-\theta_0) \end{aligned} \tag{B.6}$$

for all $\theta_0 \in (\underline{\theta}, \bar{\theta})$. Finally, we have

$$V'(\theta_0) \leq \left(\int_{\theta_0}^{\theta_L(\theta_0)} \frac{v_q(q_i(\theta_0), \theta)}{\theta_L(\theta_0) - \theta_0} f(\theta)\, \mathrm{d}\theta - \frac{v(q_i(\theta_0), \theta_0)}{q_i(\theta_0)} f(\theta_0) \right) q_i(\theta_0) \leq 0 \tag{B.7}$$

because $c'(q_i(\theta_0)) \geq \theta_L(\theta_0) > \theta_0$. □

B.2.4 Proof of Proposition 4

Proof of Proposition 4. First, I show that the point θ_0 at which conditions (S) and (C) hold coincide with the optimal cutoff that satisfies equation (OPT).

Lemma B.3. $r(\theta_0^-) \geq r(\theta_0^+)$ *for all* $\theta_0 \in (\underline{\theta}, \bar{\theta})$. *The equality holds if and only if* $v_{qq}(q, \theta_0) + \kappa c''(q) = 0$ *for almost every* $q \in (0, q_i(\theta_0))$.

Proof of Lemma B.3. $r(\theta_0+) = v_q(q_i(\theta_0), \theta_0) f(\theta_0) - \kappa f(\theta_0)(\theta_0 - \theta_c(\theta_0))$ for all $\theta_0 < \bar{\theta}$. $r(\theta_0-) = v_q(0, \theta_0) f(\theta_0) - \kappa f(\theta_0)\theta_0$ for all $\theta_0 > \underline{\theta}$. $r(\theta_0+) \leq r(\theta_0-)$ follows from $v_{qq}(q, \theta_0) + \kappa c''(q) \leq 0$ on $q \in (0, q_i(\theta_0))$ (because $\kappa = \inf\{-v_{qq}/c''(q)\}$); the equality holds if and only if $v_{qq}(q, \theta_0) + \kappa c''(q) = 0$ for almost every $q \in (0, q_i(\theta_0))$. □

Lemma B.4. *If* $\theta_0 > \underline{\theta}$, *then condition* (S) *implies* $r(\theta_0+) = r(\theta_0-) = A$ *and* $v_{qq}(q, \theta_0) + \kappa c''(q) = 0$ *for almost every* $q \in (0, q_i(\theta_0))$.

Proof of Lemma B.4. If $\theta_0 > \underline{\theta}$, then conditions (S) implies $r(\theta_0+) = L(\theta_0+|\theta_0) \geq L(\theta_0-|\theta_0) = r(\theta_0-)$. By Lemma B.3, we must have $r(\theta_0+) = r(\theta_0-) = A$. □

Lemma B.5. *Conditions* (S) *and* (C) *hold at* θ_0 *if and only if* θ_0 *satisfies equation* (OPT).

Proof of Lemma B.5. If $\theta_0 > \underline{\theta}$, condition (S) imply $L(\theta_0+|\theta_0) \geq L(\theta_0-|\theta_0)$, so by Lemma B.3, we have $L(\theta_0+|\theta_0) = L(\theta_0-|\theta_0) = A$ and $v_{qq}(q, \theta_0) + \kappa c''(q) = 0$ for almost every $q \in (0, q_i(\theta_0))$. Thus, $A = L(\theta_0+|\theta_0) = (\frac{v(q_i(\theta_0), \theta_0) + \kappa c(q_i(\theta_0))}{q_i(\theta_0)} - \kappa\theta_0) f(\theta_0) = \frac{v(q_i(\theta_0), \theta_0)}{q_i(\theta_0)} f(\theta_0)$, so $V'(\theta_0) = A \cdot q_i(\theta_0) - v(q_i(\theta_0), \theta_0) f(\theta_0) = 0$ (OPT). □

Then, I use the optimal control method, where the proposed multipliers are in Section A.1. Condition (S1) guarantees that $\mu(\theta) \leq 0$ on $(\theta_0^*, \theta_L(\theta_0^*)]$ (in equation (A.18)), so that $q^*(\theta) = q_i(\theta_0^*)$ is constant on $(\theta_0^*, \theta_L(\theta_0^*)]$. Further, condition (S1) implies that $\mu(\theta_L(\theta_0^*)) = 0$ (so μ is continuous at $\theta_L(\theta_0^*)$), and thus it is compatible with $q(\theta)$ being strictly increasing at $\theta_L(\theta_0^*)$ (as $q(\theta) = q_f(\theta)$ on $[\theta_L(\theta_0^*), \bar{\theta}]$). Condition (S2) guarantees that $\mu(\theta) \leq 0$ on $(\underline{\theta}, \theta_0^*]$, so that $q^*(\theta) = 0$ is constant on $(\underline{\theta}, \theta_0^*]$.

Sufficient condition (concavity) requires that $\Gamma + \kappa F$ is increasing. Condition (C) implies that $\Gamma + \kappa F$ is increasing on $(\theta_L(\theta_0^*), \bar{\theta}]$ where $q^*(\theta) = q_f(\theta)$. The jumps of Γ are nonnegative at $\theta_L(\theta_0^*)$ by condition (S1) if $\theta_L(\theta_0^*) < \bar{\theta}$ (by $A \geq 0$ if $\theta_L(\theta_0^*) = \bar{\theta}$) and at $\bar{\theta}$ by Assumption 1.

By Lemma B.5, θ_0^* is the optimal cutoff that satisfies the switching condition

$$H(\theta_0+) = v(q_i(\theta_0), \theta_0)f(\theta_0) + \Gamma(\theta_0+)q_i(\theta_0) = H(\theta_0-) = 0,$$

where $\Gamma(\theta_0+) = -A$. □

B.2.5 Proof of Proposition 5

To characterize the sufficient conditions, define

$$r_j(\theta|q) = v_q(q, \theta)f(\theta) - \kappa f(\theta)(\theta - c'(q)) - \kappa(F(\theta) - F(\theta_{j-1})). \tag{B.8}$$

Abusing notations, for a given allocation $q(\theta)$, define $r_j(\theta) = r_j(\theta|q(\theta))$.

Proof sketch. Condition (S-j) implies that $\mu \leq 0$ in the pooling regions and $\mu = 0$ at their intersections with fully revealing regions. Condition (C-j) guarantees the concavity in the fully revealing regions. □

B.3 Proofs of Section 5

B.3.1 Proof of Lemma 7

Proof of Lemma 7. Fully revealing ($q = q_f(\theta)$) implies

$$\Gamma(\theta) = -v_q(q_f(\theta), \theta)f(\theta),$$

and

$$\Lambda(\theta) = -[\theta\Gamma(\theta)]'/f(\theta) = \frac{v_{qq}(q_f(\theta), \theta)}{c''(q_f(\theta))}\theta + v_{q\theta}(q_f(\theta), \theta)\theta + v_q(q_f(\theta), \theta)[1 + \theta f'(\theta)/f(\theta)], \tag{B.9}$$

which must be decreasing because the Lagrangian multiplier on D is $\lambda(\theta) = -\Lambda'(\theta) \geq 0$. □

B.3.2 Proof of Proposition 7

Proof of Proposition 7. Lemma 7 covers the fully revealing region.

In the pooling regions, the optimal deterministic rating prescribes $\Gamma(\theta) = -A_j - \kappa(F(\theta) - F(\theta_j))$, so $\Lambda(\theta) = -[\theta\Gamma(\theta)]'/f(\theta) = A_j/f(\theta) + \kappa\theta + \kappa(F(\theta) - F(\theta_j))/f(\theta)$. In the case of lower censorship or pass/fail, if $(\theta_0^*, \theta_L(\theta_0^*)) = (\underline{\theta}, \bar{\theta})$ (i.e., bunching without exclusion) where θ_0^* is given by equation (OPT), one can propose $A = 0$ in the multiplier Γ. Thus, in lower censorship or pass/fail, $\tilde{A}(\theta_0) = \frac{F(\theta_L(\theta_0)) - F(\theta_0)}{\theta_c(\theta_0) - \theta_0} \cdot \mathbf{1}[(\theta_0, \theta_L(\theta_0)) \neq (\underline{\theta}, \bar{\theta})]$. □

B.3.3 Proof of Corollary 10.1

Proof of Corollary 10.1. $q_f(\theta)$ is convex if and only if $\theta / c'(q_f(\theta))$ is increasing in θ. For the second term of $J(\theta)$, $J_2(\theta) = -\frac{\int_\theta^{\bar{\theta}} 1/c'(q_f(x))\,\mathrm{d}F(x)}{f(\theta)}$, is increasing in θ for sufficiently large $\theta_L < \bar{\theta}$ because $J_2'(\theta) = \frac{c'(q_f(\theta))f(\theta) + f'(\theta)\int_\theta^{\bar{\theta}} 1/c'(q_f(x))\,\mathrm{d}F(x)}{f(\theta)^2}$. □

B.3.4 Proof of Proposition 11

Proof of Proposition 11. Substituting $\sigma(\theta)P = \sigma(\theta)[w(\theta) - c(q(\theta), \theta)] + (1 - \sigma(\theta))w_\varnothing - U(\theta)$ and $w_\varnothing = \mathbf{E}[\theta \mid \theta \leq \theta_0]$ into the objective, we have

$$
\begin{aligned}
&\int_{\underline{\theta}}^{\bar{\theta}} \sigma(\theta)\left[(1-\rho)q(\theta) + \rho(w(\theta) - c(q(\theta), \theta))\right] + \int_{\underline{\theta}}^{\theta_0} \theta\,\mathrm{d}F(\theta) - \rho U(\theta)\,\mathrm{d}F(\theta) \\
=& \int_{\underline{\theta}}^{\bar{\theta}} \sigma(\theta)\left[(1-\rho)q(\theta) + \rho(\theta - c(q(\theta), \theta)) + \rho\frac{1 - F(\theta)}{f(\theta)} c_\theta(q(\theta), \theta) - \theta\right] + \mathbf{E}[\theta]\,\mathrm{d}F(\theta)
\end{aligned}
$$

Pointwise maximization yields the first-order condition. The solution satisfies (MPS') and (BP') if $w'(\theta) = c_q(q^*(\theta), \theta)q^{*\prime}(\theta) \leq 1$, which holds if ρ is sufficiently large or $c(q, \theta)$ is sufficiently convex in q. □

Appendix C Comparison with Amador and Bagwell (2022)

For comparison purposes, I characterize sufficient conditions for lower censorship à la Amador and Bagwell (2022, henceforth AB).

Truncated problem. They first fix a cutoff θ_0 and look at the truncated problem for $\theta \geq \theta_0$. Define

$$
G(\theta|\theta_0) = \frac{1}{\theta_L(\theta_0) - \theta} \int_\theta^{\theta_L(\theta_0)} v_q(\tilde{\theta}, q_i(\theta_0)) f(\tilde{\theta}) d\tilde{\theta} - \kappa \frac{\theta - c'(q_i(\theta_0))}{\theta_L(\theta_0) - \theta}(1 - F(\theta)) - \kappa(1 - F(\theta_0)),\ \forall \theta \in [\theta_0, \theta_L(\theta_0)]. \tag{C.1}
$$

AB's Proposition 1 proposes the following two conditions.

Condition (AB(i)). $G(\theta|\theta_0) \leq G(\theta_0|\theta_0)$ for all $\theta \in [\theta_0, \theta_L(\theta_0)]$.

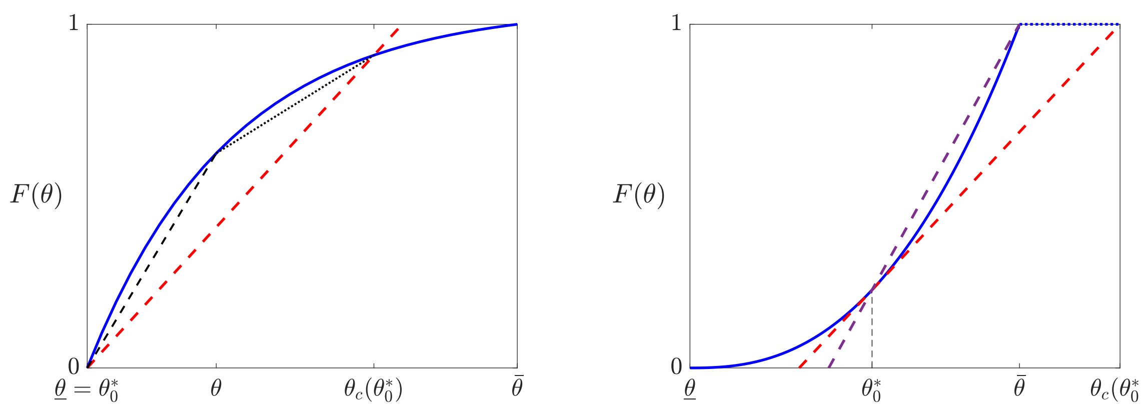
Condition (AB(ii)). $v_q(\theta, q_f(\theta))f(\theta) - \kappa F(\theta)$ is decreasing in θ on $(\theta_L(\theta_0), \bar{\theta}]$.

Observation C.1. If $r(\theta) = v_q(\theta, q_i(\theta_0))f(\theta) - \kappa(\theta - c'(q_i(\theta_0)))f(\theta) - \kappa(F(\theta) - F(\theta_0))$ is decreasing on $[\theta_0, \theta_L(\theta_0)]$ (G'), then $G(\theta|\theta_0)$ is decreasing on $\theta \in [\theta_0, \theta_L(\theta_0)]$, and condition AB(i) holds.

Condition AB(ii) is exactly the same as condition (C). For Condition AB(i), recall that condition (S) can be decomposed into conditions (S1) and (S2) on the pooling regions and exclusion regions, respectively. Condition (S2) has no counterpart in AB's conditions because they focus on the truncated problem for $\theta \geq \theta_0$. The following observations show that (S1) is weaker than AB(i).

Observation C.2. Condition AB(i) is equivalent to $L(\theta|\theta_0) \geq L(\theta_L(\theta_0)|\theta_0)$ for all $\theta \in [\theta_0, \theta_L(\theta_0)]$.

Observation C.3. If $\theta_c(\theta_0) = \theta_L(\theta_0) \leq \bar{\theta}$, then condition AB(i) is equivalent to condition (S1). In general, condition AB(i) is stronger than condition (S1) because $\theta_c(\theta_0) \geq \theta_L(\theta_0) = \min\{\theta_c(\theta_0), \bar{\theta}\}$.



(a) Satisfies both AB(i) and (S1)

(b) Satisfies (S1) but violates AB(i)

Figure C.1: Graphic Illustration of Conditions AB(i) versus Condition (S1)

For example, if $r(\theta) = f(\theta)$, Figure C.1 illustrates conditions AB(i) and (S1). In the left panel, the red dashed line represents $L(\theta_0|\theta_c(\theta_0))$ and $L(\theta_0|\theta_L(\theta_0)) = G(\theta_0|\theta_0)$. They coincide because $\theta_c(\theta_0) \leq \bar{\theta}$ (and hence $\theta_c(\theta_0) = \theta_L(\theta_0)$). For a fixed $\theta \in [\theta_0, \theta_L(\theta_0)]$, the black dashed line represents $L(\theta|\theta_0)$, while the black dotted line represents $G(\theta|\theta_0)$; the former has a higher slope than the red dashed line if and only if the latter has a lower slope than the red line. Thus, condition AB(i) and condition (S1) are equivalent if $\theta_c(\theta_0) \leq \bar{\theta}$.

In the right panel, the purple dashed line represents $L(\theta_0|\theta_L(\theta_0))$, while the red dashed line represents $L(\theta_0|\theta_c(\theta_0))$. Contrary to the previous case, because $\theta_c(\theta_0) > \bar{\theta}$ (e.g., if f is increasing), f satisfies condition (S1) but violates condition AB(i).

On the technical side, the differences between condition AB(i) and condition (S1) is because I propose a smaller multiplier A. The multiplier à la Amador and Bagwell (2022), denoted by A_{AB}, is[81]

$$\begin{aligned} A_{\text{AB}} &\equiv \frac{1}{\theta_L(\theta_0)-\theta_0}\int_{\theta_0}^{\theta_L(\theta_0)} [v_q(q_i(\theta_0),\theta)f(\theta)-\kappa f(\theta)(\theta-c'(q_i(\theta_0)))-\kappa(F(\theta)-F(\theta_0))]\,\mathrm{d}\theta \\ &= \frac{1}{\theta_L(\theta_0)-\theta_0}\left[\int_{\theta_0}^{\theta_L(\theta_0)} v_q(q_i(\theta_0),\theta)f(\theta)\,\mathrm{d}\theta-\kappa(\theta_L(\theta_0)-c'(q_i(\theta_0)))(1-F(\theta_0))\right] = G(\theta_0|\theta_0), \end{aligned}$$

while the multiplier A that I propose is

$$A = \frac{1}{c'(q_i(\theta_0))-\theta_0}\int_{\theta_0}^{\theta_L(\theta_0)} v_q(q_i(\theta_0),\theta)\,\mathrm{d}F(\theta) \le A_{\text{AB}}, \tag{C.2}$$

where the equality holds if and only if $\theta_L(\theta_0) = c'(q_i(\theta_0))$ (or equivalently, $c'(q_i(\theta_0)) \le \bar{\theta}$). Consequently, their multiplier A_{AB} forces a fully revealing region.

Global problem. Then, for global optimality, AB's Proposition 2 requires the two conditions in the truncated problem to hold for all $\theta_0 \in [\underline{\theta}, \bar{\theta})$.

Proposition C.1 (Amador and Bagwell, 2022 Propositions 1 and 2)**.** *Under Assumption 1, if conditions AB(i) and AB(ii) hold for all $\theta_0 \in [\underline{\theta}, \bar{\theta})$, the optimal deterministic rating is lower censorship (without exclusion).*

Proof. In the spirit of Amador and Bagwell (2022), fix $\theta_0 \in [\underline{\theta}, \bar{\theta})$ and look at the truncated problem for $\theta \ge \theta_0$. Because condition AB(i) implies condition (S1) with $\theta_c(\theta_0) \le \bar{\theta}$, while condition AB(ii) is the same as condition (C), Proposition 4.2.4 implies the optimal quality scheme (in the truncated problem) is

$$q(\theta) = \begin{cases} q_i(\theta_0), & \text{if } \theta \in [\theta_0, \theta_L(\theta_0)) \\ q_f(\theta), & \text{if } \theta \in [\theta_L(\theta_0), \bar{\theta}]. \end{cases} \tag{C.3}$$

Because conditions AB(i) and AB(ii) hold for all $\theta_0 \in [\underline{\theta}, \bar{\theta})$, they hold at $\theta_0 = \underline{\theta}$ in particular, so the optimal deterministic rating scheme is lower censorship with cutoff

[81]The trick used in the exchange of integrals is that we have either $\theta_L(\theta_0) = \bar{\theta}$ or $q_i(\theta_0) = q_f(\theta_L(\theta_0))$ (or both), so $(\theta_L(\theta_0) - c'(q_i(\theta_0)))(1-F(\theta_L(\theta_0))) = 0$ always holds.

$\theta_0^* = \underline{\theta}$. □

Remark 10. Condition AB(i) is stronger than condition (S1) because $\theta_c(\theta_0) \geq \theta_L(\theta_0) = \min\{\theta_c(\theta_0), \bar{\theta}\}$. Consequently, condition AB(i) implies a fully revealing region (and thus disallows pass/fail tests in general) by ruling out the possibility that $\theta_c(\theta_0) > \bar{\theta}$ (e.g., when $r(\theta)$ is increasing).

Remark 11. AB's Proposition 2 requires conditions AB(i) and AB(ii) to hold *for all* $\theta_0 \in [\underline{\theta}, \bar{\theta})$. In principle, the conditions need not hold at exclusion levels θ_0 that are dominated (e.g., $\theta_0 = \bar{\theta}$). As noted in Observation 4, this rules out the possibility of exclusion (and pass/fail tests in general).[82]

Appendix D Weaker Sufficient Conditions for Nonlinear Delegation

Lemma B.3 implies that conditions (S1) and (S2) may not hold simultaneously at $\theta_0^* > \underline{\theta}$ for nonlinear delegation. This calls for weaker sufficient conditions.

To this end, I replace κ by $k(q,\theta) \equiv -v_{qq}(q,\theta)/c''(q)$. Accordingly, the term $\kappa F(\theta)$ in previous analyses (e.g., condition (C) and the multipliers) should be replaced by $\int_{\underline{\theta}}^{\theta} k(q,\theta)\,\mathrm{d}F(\theta)$. For convenience, define $G(q,\theta) = \int_{\underline{\theta}}^{\theta} k(q,\theta)\,\mathrm{d}F(\theta)/\kappa \geq F(\theta)$ and $g(q,\theta) = k(q,\theta)f(\theta)/\kappa$. Then, it suffices to replace $F(\theta)$ and $f(\theta)$ by $G(q,\theta)$ and $g(q,\theta)$ in the equations above, respectively.

In particular, the key expressions are redefined (with a subscript w to distinguish them from the original ones) as

$$\Gamma_w(\theta) = -A - \kappa(G(q,\theta) - G(q,\theta_0)), \text{ if } \theta \in [\underline{\theta}, \theta_L(\theta_0)]. \tag{D.1}$$

$$r_w(\theta) = \begin{cases} v_q(0,\theta)f(\theta) - \kappa g(0,\theta)\theta - \kappa(G(0,\theta) - G(0,\theta_0)), & \text{if } \theta \in [\underline{\theta}, \theta_0) \\ v_q(q_i(\theta_0),\theta)f(\theta) - \kappa g(q_i(\theta_0),\theta)(\theta - \theta_c(\theta_0)) - \kappa(G(q_i(\theta_0),\theta) - F(\theta_0)), & \text{if } \theta \in [\theta_0, \theta_c(\theta_0)) \\ v_q(q_f(\theta),\theta)f(\theta) - \kappa(F(\theta) - F(\theta_0)), & \text{if } \theta \in [\theta_L(\theta_0), \bar{\theta}]. \end{cases} \tag{D.2}$$

[82]The price-cap allocation in AB still has exclusion because they assume a fixed production cost.

and $R_w(\theta) = \int_{\underline{\theta}}^{\theta} r_w(\tilde{\theta})\,\mathrm{d}\tilde{\theta}$.

$$L_w(\theta|\theta_0) = \frac{R(\theta_0) - R(\theta)}{\theta_0 - \theta}$$
$$= \begin{cases} \frac{1}{\theta_0 - \theta} \int_{\theta}^{\theta_0} [v_q(0, \tilde{\theta}) f(\tilde{\theta}) - \kappa g(0, \tilde{\theta})\tilde{\theta} - \kappa(G(0, \theta) - G(0, \theta_0))] d\tilde{\theta}, & \text{if } \theta \in [\underline{\theta}, \theta_0) \\ \frac{1}{\theta - \theta_0} \int_{\theta_0}^{\theta} [v_q(q_0, \tilde{\theta}) f(\tilde{\theta}) - \kappa g(q_0, \tilde{\theta})(\tilde{\theta} - c'(q_0)) - \kappa(G(q_0, \theta) - G(q_0, \theta_0))]\,\mathrm{d}\tilde{\theta}, & \text{if } \theta \in (\theta_0, \theta_c(\theta_0)] \end{cases}$$

where $q_0 = q_i(\theta_0)$.

Accordingly, Conditions (S*) (including (S1*) and (S2*)) are weaker due to the redefined $r_w(\theta)$ and $L_w(\theta|\theta_0)$ functions.

Condition (S*). $\int_{\theta_0}^{\theta} r_w(\tilde{\theta}) d\tilde{\theta} \geq A \cdot (\theta - \theta_0)$ for all $\theta \in [\underline{\theta}, \theta_L(\theta_0)]$, with equality at $\theta = \theta_c(\theta_0)$.

Condition (S1*). $L_w(\theta|\theta_0) \geq L_w(\theta_c(\theta_0)|\theta_0) = A$ for all $\theta \in (\theta_0, \theta_L(\theta_0)]$.

Condition (S2*). $L_w(\theta|\theta_0) \leq L_w(\theta_c(\theta_0)|\theta_0) = A$ for all $\theta \in [\underline{\theta}, \theta_0)$.

Condition (C*). $r_w(\theta) = v_q(q_f(\theta), \theta) f(\theta) - \kappa G(q_f(\theta), \theta)$ is decreasing in θ on $(\theta_L(\theta_0), \bar{\theta}]$.

Observation D.1. Condition (C) implies (C*).

The weakened conditions (S1*) and (S2*) can now hold simultaneously at $\theta_0^* > \underline{\theta}$ even if $v_{qq}(q, \theta_0) + \kappa c''(q) \neq 0$ for some $q \in (0, q_i(\theta_0))$.

Lemma D.1. *Conditions (S1*) and (S2*) can hold simultaneously at $\theta_0^* > \underline{\theta}$ given by equation (OPT) if and only if v_q is more convex than $c'(q)$, i.e., $y_{qqq}/v_{qq} \geq c'''(q)/c''(q)$.*

Proof. If $-y_{qqq}/v_{qq} \leq -c'''(q)/c''(q)$, then $k'(q) = -\frac{c''(q)y_{qqq} - c'''(q)v_{qq}}{c''(q)^2} \geq 0$, and therefore $z(q) = v_q(q, \theta_0) - k(q, \theta_0)(\theta_0 - c'(q_0))$ is increasing in q. Thus, $r_w(\theta_0+) \geq r_w(\theta_0-)$ (contrary to Lemma B.3). □

Proposition D.1. *The optimal deterministic rating scheme*

- *is lower censorship with cutoff θ_0^* if conditions (S*) and (C*) are satisfied at θ_0^*;*
- *is pass/fail if conditions (S*) is satisfied at θ_0^* such that $\theta_c(\theta_0^*) \geq \bar{\theta}$;*
- *has no exclusion (and is fully revealing if $\underline{\theta} = 0$) if conditions (S) and (C) are satisfied at $\theta_0^* = \underline{\theta}$.*

Appendix E Optimal Ratings with Transfers

E.1 Type-Contingent Transfers

In this subsection, I consider a contingent transfer $t(\theta) \in \mathbb{R}$ from the agent to the principal ($t(\theta) < 0$ denotes a net transfer from the principal to the agent). I will show that, as soon as the envelope equation and Bayesian plausibility $\mathbf{E}[w(\theta)] = \mathbf{E}[q(\theta)]$ is substituted into the principal's objective, the problem reduces to a classical mechanism design problem with transfers (e.g., Baron and Myerson, 1982; Laffont and Tirole, 1993, Chapter 1).

Assume the shadow cost of transfers is $\lambda \geq 0$, which can also be interpreted as the principal's weight of transfers $t(\theta)$ relative to $v(q, \theta)$ in her objective. In an alternative case where the principal is subject to the weak budget-balance constraint $\mathbf{E}[t(\theta)] \geq 0$, the constant $\lambda \geq 0$ can be interpreted as the Lagrangian multiplier on the constraint.[83]

Focus on a feasible direct mechanism $(q(\theta), w(\theta), t(\theta))$, where $w(\theta) = \mathbf{E}_{s\sim\pi(q(\theta))}[\mathbf{E}[q|s]]$. The principal's problem becomes

$$\max_{q,w,t} \int_{\underline{\theta}}^{\bar{\theta}} v(q(\theta), \theta) + (1+\lambda)t(\theta)\,\mathrm{d}F(\theta) \tag{E.1}$$

subject to (IC), (IR), (MPS), and (BP). With transfers, the agent's utility in (IC) and (IR) becomes $U(\hat{\theta}|\theta) = w(\hat{\theta}) - c(q(\hat{\theta}), \theta) - t(\hat{\theta})$, so the envelope condition is given by

$$w(\theta) - c(q(\theta), \theta) - t(\theta) = -\int_{\underline{\theta}}^{\theta} c_\theta(q(x), x)dx \tag{E.2}$$

Because we have a contingent transfer $t(\theta)$, the standard method applies—i.e., substituting the envelope equation to the objective and then doing pointwise maximization. By $\mathbf{E}[w(\theta)] = \mathbf{E}[q(\theta)]$ (BP), we have

$$c_q(q^*(\theta), \theta) = 1 + \frac{1}{1+\lambda} v_q(q^*(\theta), \theta) + \frac{1-F(\theta)}{f(\theta)} c_{\theta q}(q^*(\theta), \theta) \tag{E.3}$$

which satisfies

$$c_q(q^*(\theta), \theta) < c_q(q^{FB}(\theta), \theta) = 1 + v_q(q^{FB}(\theta), \theta). \tag{E.4}$$

Thus, $q^*(\theta) < q^{FB}(\theta)$. There are two sources of distortions: $\frac{1-F(\theta)}{f(\theta)} c_{\theta q}(q, \theta) < 0$ and $\frac{1}{1+\lambda} \leq 1$ (because $\lambda \geq 0$).

Lemma E.1. *Two pairs of rating schemes with contingent transfers* $\{\pi_1(\theta), t_1(\theta)\}$ *and*

[83] In this case, λ is endogenously determined by the constraint, whereas in the former case, λ is exogenously given.

$\{\pi_2(\theta), t_2(\theta)\}$ *induce the same quality scheme* $q(\theta)$ *if* $w_1(\theta) - t_1(\theta) = w_2(\theta) - t_2(\theta)$*, where* $w_i(\theta) = \hat{w}_i(q(\theta)) \equiv \mathbf{E}_{s\sim\pi_i(q(\theta))}[\mathbf{E}[q|s]]$ *is induced by* π_i.

With contingent transfers $t(\theta)$, the choice of ratings π (or $w(\theta)$) no longer matters because $t(\theta)$ can be used to provide incentives through redistribution in place of $w(\theta)$. In other words, *any* rating π can be optimal as long as the transfer $t(\theta)$ is calibrated according to the induced $w(\theta)$ such that (IC) as in equation (E.2) is satisfied. Thus, it is without loss to consider a fully revealing test $\bar{\pi}$ that induces $\hat{w}(q) = q$, in which case the optimal transfer is

$$t^*(\theta) = q^*(\theta) - c(q^*(\theta), \theta) + \int_{\underline{\theta}}^{\theta} c_\theta(q^*(x), x)dx \tag{E.5}$$

Proposition E.1. *With contingent transfer, the optimal mechanism is a fully revealing test and a transfer* $t^*(\theta)$ *that satisfies the envelope equation* (E.2).

In the extreme case where $\lambda \to \infty$, that is, the principal cares only about the transfer (i.e., certification fee), we have $c_q(q^*(\theta), \theta) = 1 + \frac{1-F(\theta)}{f(\theta)} c_{\theta q}(q^*(\theta), \theta)$ (Albano and Lizzeri, 2001).

Of course, if one imposes some restrictions on the contingent transfers, such as the strong budget balance ($\mathbf{E}[t(\theta)] = 0$) or one-sided transfers ($t(\theta) \geq 0$ or $t(\theta) \leq 0$), the choice of ratings will matter again. I shall focus on the restriction of constant transfers.

E.2 Constant Transfers in Deterministic Ratings

With a constant transfer $t(\theta) = t \in \mathbb{R}$ from the agent to the principal ($t(\theta) < 0$ denotes a net transfer from the principal to the agent), the indifference quality $q_i(\theta; t)$ is now given by

$$-t + q_i(\theta; t) - c(q_i(\theta; t))/\theta = 0 \tag{E.6}$$

Therefore,

$$\frac{\mathrm{d}q_i(\theta; t)}{\mathrm{d}t} = -\frac{1}{c'(q_i(\theta; t))/\theta - 1} < 0. \tag{E.7}$$

It is strictly negative because $c'(q_i(\theta; t)) > c(q_i(\theta; t))/q_i(\theta; t) > c(q_i(\theta; t))/[q_i(\theta; t) - t] = \theta$. The second-order derivative is $\frac{\mathrm{d}^2 q_i(\theta;t)}{\mathrm{d}t^2} = \frac{c''(q_i(\theta;t))/\theta}{c'(q_i(\theta;t))/\theta - 1} \frac{\mathrm{d}q_i(\theta;t)}{\mathrm{d}t} < 0$. Thus,

$$\frac{\mathrm{d}q_i(\theta; t)}{\mathrm{d}t} + 1 \geq 0 \iff c'(q_i(\theta; t)) \geq 2\theta. \tag{E.8}$$

Example E.1. Assume quadratic cost $c(q) = q^2/2$. Then, $q_i(\theta; t) = \theta + \sqrt{\theta^2 - t\theta} \geq 2\theta \iff$

$t \leq 0 \iff \frac{dq_i(\theta;t)}{dt} + 1 \geq 0$. Assume the principal assigns the same weight for the quality q and transfers t, that is, the objective is $q + t$. Then, for each type θ for whom the (IR) binds and chooses $q_i(\theta; t)$, the optimal transfer (fee or subsidy) t^* is given by $c'(q_i(\theta; t^*)) = 2\theta$, which is zero given the quadratic cost. For more convex costs, $t^* > 0$ is a fee. For more concave costs, $t^* < 0$ is a subsidy.

In general, a subsidy (fee) has three effects on the principal's expected payoff:

(i) increasing (decreasing) the quality $q_i(\theta; t)$ for all $\theta \in [\theta_0, \theta_L(\theta_0)]$,

(ii) increasing the total subsidy (fee) to all $\theta \geq \theta_0$, and

(iii) changing the optimal cutoff type θ_0^* and thus $\theta_L(\theta_0^*)$.

While the direction of the third effect is ambiguous, a fee ($t > 0$) will lead to higher *minimum* exclusion. Define the minimum exclusion $\theta_{\min}(t)$ by

$$q_f(\theta_{\min}; t) = q_i(\theta_{\min}; t) \iff -t + q_f(\theta_{\min}; t) - c(q_f(\theta_{\min}; t))/\theta_{\min} = 0, \tag{E.9}$$

where $c'(q_f(\theta; t)) = \theta$. When there is no transfer, $\theta_{\min}(0) = 0$ because $c(q)/q \geq 0$ by the convexity of $c(q)$ and $c(0) = c'(0) = 0$ (with equality if and only if $q = 0$). Thus, $\theta_{\min}(t) \geq 0$ if and only if $t \geq 0$. Moreover,

$$\theta'_{\min}(t) = \frac{\theta_{\min}^2}{c(q_f(\theta_{\min}; t))} > 0. \tag{E.10}$$

Example (Quadratic cost)**.** When $c(q) = q^2/2$, the minimum exclusion is $\theta_{\min}(t) = 2t$.

The principal can design a minimum standard to induce a higher exclusion $\theta_0 \geq \theta_{\min}$ but not any lower. Assume the principal's objective is $(1-\alpha)v(q,\theta) + \alpha t$. The principal's expected payoff is

$$V(\theta_0; t) = \int_{\theta_0}^{\theta_L(\theta_0)} ((1-\alpha)v(q_i(\theta_0), \theta) + \alpha t)\, \mathrm{d}F(\theta) + \int_{\theta_L(\theta_0)}^{\bar{\theta}} ((1-\alpha)v(q_f(\theta), \theta) + \alpha t)\, \mathrm{d}F(\theta). \tag{E.11}$$

The optimal t and $\theta_0^*(t)$ are given by

$$\max_{t \in \mathbb{R},\, \theta_0 \in \Theta} V(\theta_0; t) \quad \text{s.t. } \theta_0 \geq \theta_{\min}(t). \tag{E.12}$$

In the quality maximization case $v(q, \theta) = q$, this means that the optimal rating can have exclusion (and thus can be pass/fail) even if the $f(\theta)$ is decreasing.

Quality Maximization. Assume $v(q, \theta) = q$ so that the principal maximizes $(1-\alpha)q + \alpha t$. By the envelope theorem,

$$\begin{aligned}\frac{\mathrm{d}V(\theta_0^*(t);t)}{\mathrm{d}t} &= \int_{\theta_0}^{\theta_L(\theta_0^*)} (1-\alpha)v_q(q_i(\theta_0^*;t),\theta)\frac{\mathrm{d}q_i(\theta_0^*;t)}{\mathrm{d}t}\,\mathrm{d}F(\theta) + \alpha(1-F(\theta_0^*)) - \lambda\theta_{\min}'(t) \\ &= (1-\alpha)(F(\theta_L(\theta_0^*)) - F(\theta_0))\frac{\mathrm{d}q_i(\theta_0^*;t)}{\mathrm{d}t} + \alpha(1-F(\theta_0^*)) - \lambda\theta_{\min}'(t) \\ &= -(1-\alpha)(F(\theta_L(\theta_0^*)) - F(\theta_0))\frac{\theta}{c'(q_i(\theta_0^*;t)) - \theta} + \alpha(1-F(\theta_0^*)) - \lambda\theta_{\min}'(t)\end{aligned} \tag{E.13}$$

In general, if the optimal test is lower censorship, a fee/subsidy t can potentially change the structure of lower censorship—it will change θ_0^* and thus $\theta_L(\theta_0^*)$.

Example E.2 (Quadratic cost)**.** Assume $c(q) = q^2/2$. Assume the constraint on $\theta_0 \geq 2t$ is not binding ($\lambda = 0$), either because t is a subsidy ($t < 0$) or a small fee. If the optimal test is pass/fail, then the first-order condition $c'(q_i(\theta;t^*)) = \theta/\alpha$ implies the optimal transfer is $t^* = (2\alpha - 1)\theta/\alpha^2$. When $\alpha \geq 1/2$ ($\alpha \leq 1/2$), that is, the principal puts more (less) weight on transfer than the quality, the optimal transfer $t^* \geq 0$ ($t^* \leq 0$) is a fee (subsidy).

Monopoly Certifier. Assume the principal maximizes the expected certification fee ($\alpha = 1$), i.e., $V(\theta_0, t) = t(1 - F(\theta_0))$. Because the principal cannot benefit from any higher exclusion, $\theta_0(t) = \theta_{\min}(t)$ must be binding; therefore, $\theta_0'(t) = \frac{\theta_{\min}^2}{c(q_f(\theta_{\min};t))}$. Thus, the optimal fee t^* is given by

$$\frac{\mathrm{d}V(\theta_0,t)}{\mathrm{d}t} = (1 - F(\theta_0)) - t^* f(\theta_0)\theta_0'(p^*) \leq 0, \quad \frac{\mathrm{d}V(\theta_0,t)}{\mathrm{d}t}\cdot(\theta_0(t^*) - \underline{\theta}) = 0. \tag{E.14}$$

Example E.3 (Quadratic cost)**.** If $c(q) = q^2/2$, then $\theta_0(t) = 2t$. Thus, the optimal exclusion is given by $\frac{1-F(\theta_0^*)}{f(\theta_0^*)} = \theta_0^*$ and the optimal transfer (certification fee) is $t^* = \theta_0^*/2$.

E.3 Constant Transfers in General Ratings

Denote the dummy decision variable that the agent of type θ takes the test by $\sigma(\theta) = \mathbf{1}[\theta \text{ takes the test}]$. Assume that the principal's objective is $v(q, \theta) + \sigma(\theta)\alpha P$ where $P \geq 0$ is the testing fee. The principal's problem is

$$\max_{q,w,P,\sigma} \int_{\underline{\theta}}^{\bar{\theta}} ((1-\alpha)v(q(\theta),\theta) + \alpha P\sigma(\theta))\,\mathrm{d}F(\theta) \tag{E.15}$$

subject to

$$\sigma(\theta)[w(\theta) - c(q(\theta), \theta) - P] = -\int_{\underline{\theta}}^{\theta} \sigma(x) c_\theta(q(x), x)\, \mathrm{d}x \qquad \text{(IC-Env)} \tag{E.16}$$

$$q(\theta) \text{ increasing} \qquad \text{(IC-Mon)} \tag{E.17}$$

$$\sigma(\theta) \in \{0, 1\} \text{ increasing} \tag{E.18}$$

$$w(\theta) - c(q(\theta), \theta) - P \geq 0 \qquad \text{(IR)} \tag{E.19}$$

$$\int_{\underline{\theta}}^{\bar{\theta}} \sigma(\theta) w(\theta)\, \mathrm{d}F(\theta) = \int_{\underline{\theta}}^{\bar{\theta}} \sigma(\theta) q(\theta)\, \mathrm{d}F(\theta) \qquad \text{(BP)} \tag{E.20}$$

$$\int_{\underline{\theta}}^{\theta'} \sigma(\theta) w(\theta')\, \mathrm{d}F(\theta') \geq \int_{\underline{\theta}}^{\theta} \sigma(\theta') q(\theta)\, \mathrm{d}F(\theta'),\ \forall \theta \in \Theta \qquad \text{(MPS)} \tag{E.21}$$

Lemma E.2. *If $\sigma(\theta) = 0$, then $q(\theta) = 0$, and the market offers $w_\varnothing = 0$.*

Lemma E.3. *There exists a cutoff type θ_0 such that $\sigma(\theta) = 1$ if and only if $\theta \geq \theta_0$.*

Define $D(\theta) = \int_{\underline{\theta}}^{\theta} (w(\theta') - q(\theta'))\, \mathrm{d}F(\theta') \geq 0$ and $U(\theta) = -\int_{\theta_0}^{\theta} c_\theta dx$. Given the cutoff type θ_0, the principal's problem can be rewritten as

$$\max_{q,w,P} \int_{\theta_0}^{\bar{\theta}} (1-\alpha) v(q(\theta), \theta) + \alpha(w(\theta) - c(q(\theta), \theta) - U(\theta))\, \mathrm{d}F(\theta) \tag{E.22}$$

subject to

$$D(\theta) \geq 0 \text{ (MPS)} \tag{E.23}$$

$$\dot{D} = [w(\theta) - q(\theta)] f(\theta) \tag{E.24}$$

$$w(\theta) - c(q(\theta), \theta) - P = U(\theta) \tag{E.25}$$

$$\dot{U} = -c_\theta \tag{E.26}$$

$$\dot{q} = \nu \geq 0 \tag{E.27}$$

$$U(\theta) \geq 0 \tag{E.28}$$

$$D(\bar{\theta}) = 0 \text{ (BP)}, \quad U(\bar{\theta}), q(\bar{\theta}) \text{ free.} \tag{E.29}$$

Set up the Hamiltonian

$$\begin{aligned} H = {} & [v(q(\theta), \theta) + w(\theta) - c(q(\theta), \theta) - U(\theta)] f(\theta) + \gamma(\theta)[(w(\theta) - c(q(\theta), \theta) - P) - U(\theta)] \\ & + \lambda(\theta) D(\theta) + \phi(\theta) U(\theta) + \Lambda(\theta)[w(\theta) - q(\theta)] f(\theta) - \Gamma(\theta) c_\theta + \mu \nu(\theta) \end{aligned} \tag{E.30}$$

where U, q, D are state variables and w, ν are the control variable; $\lambda(\theta)$ is the Lagrangian multiplier on $D(\theta) \geq 0$ (MPS), $\gamma(\theta)$ is the Lagrangian multiplier on $U(\theta) = w(\theta) - c(q(\theta), \theta)$, Λ is the Hamiltonian multiplier on $\dot{D} = [w(\theta) - q(\theta)]f(\theta)$, and Γ is the Hamiltonian multiplier on $\dot{U} = -c_\theta$.

By the Pontryagin's maximum principle,

$$-\frac{\partial H}{\partial q} = -(v_q f - (f + \gamma)c_q - \Lambda f - \Gamma c_{\theta q}) = \dot{\mu} \tag{E.31}$$

$$-\frac{\partial H}{\partial D} = -\lambda = \dot{\Lambda} \tag{E.32}$$

$$-\frac{\partial H}{\partial U} = f + \gamma - \phi = \dot{\Gamma} \tag{E.33}$$

$$\frac{\partial H}{\partial w} = f + \gamma + \Lambda f = 0 \tag{E.34}$$

$$\frac{\partial H}{\partial \nu} = \mu \leq 0, \quad \mu\nu = 0 \tag{E.35}$$

$$\Lambda(\bar{\theta}) \text{ no condition}, \quad \Gamma(\bar{\theta}) = 0. \tag{E.36}$$

When $\alpha = 1$,

$$c_q(q(\theta), \theta) = 1/\alpha + \frac{1 - F(\theta)}{f(\theta)} c_{\theta q}(q(\theta), \theta) < 1,$$

so (MPS) does not bind, and $w(\theta) = c_q(q(\theta), \theta)q'(\theta) < q'(\theta)$ implies a noisy test.

[TBA]

Appendix F General Cost Functions

In this section, I consider general cost functions $c(q, \theta)$ that satisfies $c_q > 0$, $c_\theta < 0$, $c_{qq} > 0$, $c_{q\theta} < 0$. The assumption that zero investment has no cost can be weakened to: for any $\theta \in [\underline{\theta}, \bar{\theta}]$, there exists $\underline{q}(\theta) \in Q$ such that $c(\underline{q}(\theta), \theta) = c_q(\underline{q}(\theta), \theta) = 0$. Hence, Lemma 1 becomes

Lemma F.1 (Cf. Lemma 1)**.** *In any equilibrium, if an agent of type* θ *does not take the test, he chooses* $q = \underline{q}(\theta)$ *such that* $c(\underline{q}(\theta), \theta) = 0$*, and the market offers him* $\omega(\varnothing) = \underline{q}(\underline{\theta})$.

In particular, with a slight abuse of notation, I will also focus on a commonly used cost function $c(q, \theta) = c(q - \theta)$.[84] This specification has a realistic interpretation that a type-θ agent is endowed with quality θ and can invest effort $e \geq 0$ at the cost $c(e)$ to achieve quality $q = \theta + e$. When the market values quality q (type θ), this effort is productive (manipulative). Assume $c' > 0$, $c'' > 0$, $c''' \geq 0$, $c(0) = c'(0) = 0$.

[84]For example, in Laffont and Tirole (1993); Augias and Perez-Richet (2023); Perez-Richet and Skreta (2022)

For general cost functions, the fully revealing quality $q_f(\theta)$ is given by

$$c_q(q_f(\theta), \theta) = 1. \tag{F.1}$$

The no-effort quality $\underline{q}(\theta)$ is given by

$$c(\underline{q}(\theta), \theta) = 0. \tag{F.2}$$

The indifference quality $q_i(\theta)$ is given by

$$q_i(\theta) - c(q_i(\theta), \theta) = \underline{q}(\theta). \tag{F.3}$$

Particularly, for the specification $c(q, \theta) = c(q - \theta)$, we have

$$c'(q_f(\theta) - \theta) = 1 \implies q_f(\theta) = \theta + e_f \tag{F.4}$$

$$c(\underline{q}(\theta) - \theta) = 0 \implies \underline{q}(\theta) = \theta \tag{F.5}$$

$$q_i(\theta) - \theta = c(q_i(\theta) - \theta) \implies q_i(\theta) = \theta + e_i \tag{F.6}$$

where $e_f = c'^{-1}(1)$ and $e_i > 0$ is the unique fixed point of $c(e)$ on $\mathbb{R}_{++}$ ($e_i = c(e_i) > e_f$).

F.1 Deterministic Ratings

The principal's problem is

$$\max_{q(\theta)} \int_{\underline{\theta}}^{\bar{\theta}} v(q(\theta), \theta) \, \mathrm{d}F(\theta) \tag{F.7}$$

subject to

$$q(\theta) - c(q(\theta), \theta) \geq 0 \qquad \text{(IR)} \tag{F.8}$$

$$q(\theta) - c(q(\theta), \theta) = -\int_{\underline{\theta}}^{\theta} c_\theta(q(x), x) dx + \underline{U} \qquad \text{(IC-Env)} \tag{F.9}$$

$$q(\theta) \text{ increasing} \qquad \text{(IC-Mon)} \tag{F.10}$$

Rewrite the constraints and set up the Hamiltonian

$$H = v(q(\theta), \theta) f(\theta) + \gamma(\theta)[q(\theta) - c(q(\theta), \theta) - U(\theta)] - \Gamma(\theta) c_\theta + \mu(\theta)\nu(\theta) \tag{F.11}$$

By the Pontryagin's maximum principle,

$$-\frac{\partial H}{\partial q} = -(v_q f + \gamma(1-c_q) - \Gamma c_{\theta q}) = \dot{\mu} \tag{F.12}$$

$$-\frac{\partial H}{\partial U} = \gamma = \dot{\Gamma} \tag{F.13}$$

$$\frac{\partial H}{\partial \nu} = \mu \leq 0, \quad \mu(\theta) = 0 \text{ if } q \text{ is strictly increasing at } \theta, \tag{F.14}$$

$$\Gamma(\underline{\theta}) \leq 0, \quad \Gamma(\underline{\theta})U(\underline{\theta}) = 0 \tag{F.15}$$

$$\mu(\underline{\theta}) \leq 0, \quad \mu(\underline{\theta})q(\underline{\theta}) = 0 \tag{F.16}$$

$$\Gamma(\bar{\theta}) = 0, \quad \mu(\bar{\theta}) = 0. \tag{F.17}$$

Fully revealing $q = q_f(\theta) \implies$

$$\Gamma = \frac{v_q(q_f(\theta),\theta)}{c_{\theta q}(q_f(\theta),\theta)} f(\theta). \tag{F.18}$$

Now assume throughout the subsection that $c(q,\theta) = c(q-\theta)$ and denote $e(\theta) = q(\theta) - \theta$. Therefore,

$$q_f(\theta) = \theta + e_f \tag{F.19}$$

$$\underline{q}(\theta) = \theta \tag{F.20}$$

$$q_i(\theta) = \theta + e_i \tag{F.21}$$

where $e_i > e_f > 0$ are constants. Then, fully revealing $q = q_f(\theta) \implies$

$$\Gamma(\theta) = -\frac{v_q(q_f(\theta),\theta)}{c''(e_f)} f(\theta). \tag{F.22}$$

By Kamien and Schwartz (1971), sufficiency requires

$$v_{qq} f - \gamma c''(e(\theta)) + \Gamma c'''(e(\theta)) \leq 0 \tag{F.23}$$

which holds if $\Gamma \cdot c''(e(\theta))$ is decreasing because $v_{qq} f \leq 0$ and $\Gamma \cdot c'''(e(\theta)) \leq 0$.

[TBA]

Condition (C'). $r(\theta) \equiv v_q(q_f(\theta),\theta) f(\theta)$ is decreasing in θ on $(\theta_L(\theta_0), \bar{\theta}]$.

Proposition F.1. *If condition (C') holds at $\theta_0 = \underline{\theta}$, then the optimal deterministic rating is lower censorship without exclusion.*

Corollary F.1.1 (Quality maximization)**.** *Assume $v(q,\theta) = q$. The optimal deterministic rating is lower censorship without exclusion if and only if $f(\theta)$ is decreasing.*

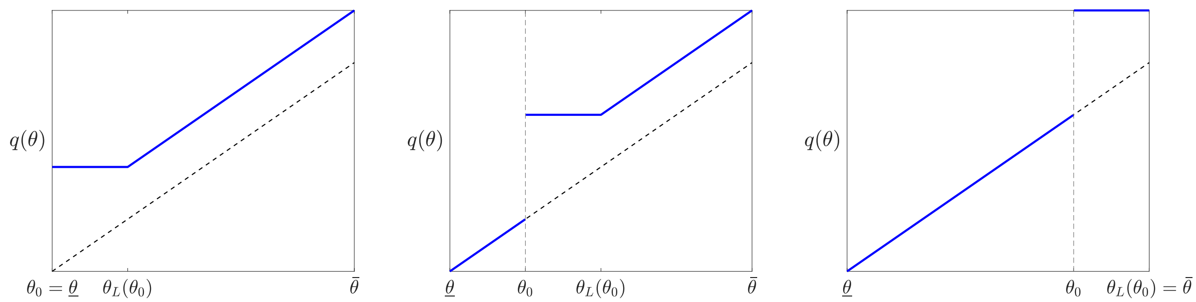


Figure F.1: $q^*(\theta)$ under lower censorship (without and with exclusion) and pass/fail

A linear-quadratic example. Assume $v(q, \theta) = q$ and $c(e) = e^2/2$. Then, $e_f = 1$ and $e_i = 2$; thus, $\underline{q}(\theta) = \theta$, $q_f(\theta) = \theta + 1$, $q_i(\theta) = \theta + 2$, and $\theta_c(\theta_0) = \theta_0 + 1$.

Note that lower censorship or pass/fail tests that lead to exclusion are never optimal because a "low-pass" minimum standard at θ_0 can always increase the quality of the types $[\max\{\theta_0 - 1, \underline{\theta}\}, \theta_0]$.

The optimal cutoff θ_0^* is given by

$$F(\theta_0^* + 1) - F(\theta_0^* - 1) \leq 2f(\theta_0^*), \quad [F(\theta_0^* + 1) - F(\theta_0^* - 1) - 2f(\theta_0^*)](\theta_0^* - \underline{\theta}) = 0. \tag{F.24}$$

If $f(\theta)$ is decreasing, $\theta_0^* = \underline{\theta}$ is optimal because $F(\theta_0 + 1) - F(\theta_0 - 1) \leq 2f(\theta_0)$. Thus, the optimal deterministic rating is lower censorship without exclusion.

If $f(\theta)$ is increasing, multiple jumps in $q(\theta)$ is optimal because $F(\theta_0 + 1) - F(\theta_0 - 1) \geq 2f(\theta_0)$ for all $\theta_0 \leq \bar{\theta} - 1$. For example, $F(\theta) = \theta^2/\bar{\theta}^2$ and $\Theta = [0, 3]$, the optimal cutoff is $\theta_0^* = 3$ and $\theta_0^* = 1$. The optimal deterministic rating has a "low-pass" minimum standard $q_0 = 2$ and a "high-pass" minimum standard $q_0 = 4$.

If $\theta \sim$ Unif $[\underline{\theta}, \bar{\theta}]$, because $F(\theta_0 + 1) - F(\theta_0 - 1) = 2f(\theta_0)$, optimal quality schemes include both forms (and many more).

F.2 General Ratings

The principal's problem is

$$\max_{q(\theta), w(\theta)} \int_{\underline{\theta}}^{\bar{\theta}} v(q(\theta), \theta) \, \mathrm{d}F(\theta) \tag{F.25}$$

subject to

$$w(\theta) - c(q(\theta),\theta) = -\int_{\underline{\theta}}^{\theta} c_\theta(q(x),x)dx + \underline{U} \qquad \text{(IC-Env)} \tag{F.26}$$

$$q(\theta) \text{ increasing} \qquad \text{(IC-Mon)} \tag{F.27}$$

$$w(\theta) - c(q(\theta),\theta) \geq 0 \qquad \text{(IR)} \tag{F.28}$$

$$\int_{\underline{\theta}}^{\bar{\theta}} w(\theta)\,\mathrm{d}F(\theta) = \int_{\underline{\theta}}^{\bar{\theta}} q(\theta)\,\mathrm{d}F(\theta) \qquad \text{(BP)} \tag{F.29}$$

$$\int_{\underline{\theta}}^{\theta} w(\theta')\,\mathrm{d}F(\theta') \geq \int_{\underline{\theta}}^{\theta} q(\theta)\,\mathrm{d}F(\theta'),\ \forall \theta \in \Theta \qquad \text{(MPS)} \tag{F.30}$$

Define $D(\theta) = \int_{\underline{\theta}}^{\theta} (w(\theta') - q(\theta'))\,\mathrm{d}F(\theta') \geq 0$ and $U(\theta) = -\int_{\underline{\theta}}^{\theta} c_\theta(q(x),x)dx + \underline{U}$. Rewrite the constraints as

$$D(\theta) \geq 0 \text{ (MPS)} \tag{F.31}$$

$$\dot{D} = [w(\theta) - q(\theta)]f(\theta) \tag{F.32}$$

$$w(\theta) - c(q(\theta),\theta) = U(\theta) \tag{F.33}$$

$$\dot{U} = -c_\theta(q(\theta),\theta) \tag{F.34}$$

$$\dot{q} = \nu \geq 0 \text{ (q increasing if $\dot{q}$ does not exist)} \tag{F.35}$$

$$U(\underline{\theta}), q(\underline{\theta}) \geq 0,\ \ D(\underline{\theta}) = 0 \tag{F.36}$$

$$U(\bar{\theta}), q(\bar{\theta}) \text{ free},\ \ D(\bar{\theta}) = 0 \text{ (BP)} \tag{F.37}$$

Set up the Hamiltonian

$$\begin{aligned} H = v(q(\theta),\theta)f(\theta) + \gamma(\theta)[w(\theta) - c(q(\theta),\theta) - U(\theta)] + \lambda(\theta)D(\theta) \\ + \Lambda(\theta)[w(\theta) - q(\theta)]f(\theta) - \Gamma(\theta)c_\theta + \mu(\theta)\nu(\theta) \end{aligned} \tag{F.38}$$

where U, q, D are state variables and w, ν are the control variable; $\lambda(\theta)$ is the Lagrangian multiplier on $D(\theta) \geq 0$ (MPS), $\gamma(\theta)$ is the Lagrangian multiplier on $U(\theta) = w(\theta) - c(q(\theta),\theta)$, Λ is the Hamiltonian multiplier on $\dot{D} = [w(\theta) - q(\theta)]f(\theta)$, and Γ is the Hamiltonian multiplier on $\dot{U} = -c_\theta(q(\theta),\theta)$.

By the Pontryagin's maximum principle,

$$-\frac{\partial H}{\partial q} = -(v_q f - \gamma c_q - \Lambda f - \Gamma c_{\theta q}) = \dot{\mu} \tag{F.39}$$

$$-\frac{\partial H}{\partial D} = -\lambda = \dot{\Lambda} \tag{F.40}$$

$$-\frac{\partial H}{\partial U} = \gamma = \dot{\Gamma} \tag{F.41}$$

$$\frac{\partial H}{\partial w} = \gamma + \Lambda f = 0 \tag{F.42}$$

$$\frac{\partial H}{\partial \nu} = \mu \leq 0, \quad \mu(\theta) = 0 \text{ if } q \text{ is strictly increasing at } \theta \tag{F.43}$$

$$\lambda(\theta) \geq 0, \quad \lambda(\theta) D(\theta) = 0 \tag{F.44}$$

$$\Gamma(\underline{\theta}) \leq 0, \quad \Gamma(\underline{\theta}) U(\underline{\theta}) = 0 \tag{F.45}$$

$$\mu(\underline{\theta}) \leq 0, \quad \mu(\underline{\theta}) q(\underline{\theta}) = 0 \tag{F.46}$$

$$\Gamma(\bar{\theta}) = 0, \quad \mu(\bar{\theta}) = 0 \tag{F.47}$$

$$\Lambda(\bar{\theta}) \text{ no condition.} \tag{F.48}$$

Thus,

$$\dot{\Gamma} = \gamma = -\Lambda(\theta) f(\theta) \tag{F.49}$$

$$\dot{\mu} = -\dot{\Gamma}(\theta)(1 - c_q) - v_q(q(\theta), \theta) f(\theta) + \Gamma(\theta) c_{\theta q} \tag{F.50}$$

$$\dot{\Lambda}(\theta) = -\lambda(\theta) \leq 0, \quad \lambda(\theta) \int_{\underline{\theta}}^{\theta} (w(\theta') - q(\theta'))\, \mathrm{d}F(\theta') = 0 \tag{F.51}$$

[TBA]

F.3 Ability Signaling

Assume the cost function is $c(q, \theta)$. Define $q_f(\theta)$ as the quality scheme under full separation, which is characterized by

$$\hat{w}(q_f(\theta)) \equiv w(\theta) = \theta, \tag{BP}$$

$$\hat{w}'(q_f(\theta)) = c_q(q_f(\theta), \theta) \tag{FOC}$$

$$\underline{\theta} - c(q_f(\underline{\theta}), \underline{\theta}) = 0 \tag{IR}$$

Together, they imply $c_q(q_f(\theta), \theta) \cdot q_f'(\theta) = 1$ and $c(q_f(\underline{\theta}), \underline{\theta}) = \underline{\theta}$. Denote

$$J(\theta|q_f) = -\frac{v_q(q_f(\theta), \theta)}{c_q(q_f(\theta), \theta)} - \frac{\int_{\theta}^{\bar{\theta}} v_q(q_f(x), x)\, \mathrm{d}F(x)}{f(\theta)} \frac{c_{qq}(q_f(\theta), \theta)/c_q(q_f(\theta), \theta) + c_{q\theta}(q_f(\theta), \theta)}{c_q(q_f(\theta), \theta)^2} \tag{F.52}$$

In the specification throughout this section, $c(q, \theta) = c(q - \theta)$, which means a type-$\theta$ agent can invest a manipulative effort $e \geq 0$ at the cost $c(e)$ to achieve quality $q = \theta + e$, where $c' > 0$, $c'' > 0$, $c''' \geq 0$, $c(0) = 0$. The effort scheme $e_f(\theta) \equiv q_f(\theta) - \theta$ under full separation

$c'(e_f(\theta)) \cdot (1 + e_f'(\theta)) = 1$, and $J(\theta|q_f)$ simplifies to

$$J(\theta|e_f) = -\frac{v_q(q_f(\theta), \theta)}{c'(e_f(\theta))} - \frac{\int_\theta^{\bar{\theta}} v_q(q_f(x), x)\, \mathrm{d}F(x)}{f(\theta)} \frac{c''(e_f(\theta))}{c'(e_f(\theta))^2} e_f'(\theta) \tag{F.53}$$

In the effort-maximizing case (i.e., $v(q, \theta) = e = q - \theta$), it further simplifies to

$$J(\theta|e_f) = -\frac{1}{c'(e_f(\theta))} - \frac{1 - F(\theta)}{f(\theta)} \frac{c''(e_f(\theta))}{c'(e_f(\theta))^2} e_f'(\theta) \tag{F.54}$$

Proposition F.2 (Cf. Proposition 10)**.** *The optimal rating induces full separation $q^*(\theta) = q_f(\theta)$ if and only if $J(\theta|q_f)$ is increasing in θ.*

Example F.1. Assume quadratic cost $c(e) = e^2/2$ and $\underline{\theta} = 0.5$; then, $e_f(\theta) = 1$ and $\hat{w}(q) = q - 1$. The effort-maximizing rating induces $e^*(\theta) = 1$ and $q^*(\theta) = \theta + 1$ (because $J(\theta|e_f) = -1$ is constant).

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