

On the Optimality of Price Caps: A Pontryagin Approach

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Abstract

I study the Amador and Bagwell (2022) model of monopolist regulation without transfers. Using the optimal control method, I provide weaker sufficient conditions for the optimality of price-cap regulation, which accommodate cases where the monopolist in the market always sets the price at the cap. For linear demand, price caps are optimal if the cost density is log-concave or decreasing. For log-convex demand functions with constant curvature (e.g., logarithmic and constant elasticity demand), price caps are optimal if the cost density is log-concave or increasing. Methodologically, I develop a sufficiency theorem for optimal control problems with monotonicity and equality constraints on state variables, which can be applied to delegation problems with or without participation constraints.

Keywords: Price caps; fixed-price regulation; delegation; optimal control.

1 Introduction

In a seminal paper, [Amador and Bagwell \(2022, henceforth AB\)](#) study the optimal regulation of a monopolist with private information about its production costs when transfers are infeasible (see also [Baron and Myerson \(1982\)](#)). Building on the cumulative Lagrangian approach widely used in the delegation literature, and extending it to incorporate a participation constraint, they derive sufficient conditions under which price-cap regulation is optimal.

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In this paper, I provide more general conditions for the optimality of price caps, which improve their result in two aspects. First, under my conditions, the optimal price-cap regulation can induce a *bang-bang* allocation where the monopolist either shuts down or always sets the price at the cap. In this case, the optimal regulation can also take the form of fixed-price regulation—i.e., a take-it-or-leave-it offer where the firm either accepts the fixed price or shuts down. This bang-bang allocation can be more realistic because, in practice, a regulated monopolist in the market rarely sets its price below the cap; in other words, price caps are usually binding for participants. By contrast, their conditions imply that there exists a flexible pricing region below the price cap, thus ruling out bang-bang allocations. Mathematically, the improvement is due to a more precise multiplier that does not rely on the existence of a flexible pricing region.

Second, as they point out in the conclusion, their sufficient conditions guarantee that the price-cap allocation is optimal *for any given level of exclusion* and are therefore more restrictive than necessary. In particular, in the absence of the fixed cost, their conditions imply that a price-cap allocation without exclusion is always optimal, thus ruling out cases in which exclusion is desirable for reasons other than saving the fixed cost. By contrast, I characterize the optimal exclusion level when there is no fixed cost and require the conditions to hold for the optimal level. Because this level also provides a lower bound on the optimal exclusion level when there is a fixed cost, the sufficient conditions also extend to the case with a fixed cost.

As an application, I consider a family of (inverse) demand functions with constant curvature of demand (see, e.g., [Aguirre et al. \(2010\)](#); [Mrázová and Neary \(2017\)](#)), including logarithmic, linear, and constant elasticity demand functions. For linear demand, price caps are optimal if the cost density is log-concave or decreasing, and fixed-price regulation (which leads to a bang-bang allocation) is optimal if the density is decreasing. For log-convex demand functions with constant curvature (e.g., logarithmic and constant elasticity demand), price caps are optimal if the cost density is log-concave or increasing.

Methodologically, I develop a sufficiency theorem for optimal control problems with a monotonicity constraint on one variable (without assuming its continuity) and an equality constraint on state variables. The sufficiency theorem naturally extends the existing Pontryagin maximum principles and sufficiency theorems in optimal control literature (e.g., [Léonard and Long \(1992\)](#); [Seierstad and Sydsæter \(1987\)](#); [Hellwig \(2008\)](#)) to address two technical issues that arise in delegation problems. The proof relies on the global theory of constrained optimization ([Luenberger, 1969](#), Chapter 8), which is also used in [Amador and Bagwell's \(2013\)](#) and [Kartik et al. \(2021\)](#). The optimal control method simplifies the (cumulative) Lagrangian method developed by [Amador et al. \(2006\)](#) and

can be applied to solve delegation problems in general (e.g. Amador and Bagwell, 2013, 2022; Guo, 2016).

In general, the results in this paper are an application of the optimal control method to delegation problems with participation constraints. Optimal delegation with participation constraints (see also Kartik et al. (2021); Kolotilin and Zapechelnyuk (2025); Xiao (2024)) is more complex than that without participation constraints because the optimal allocation is typically discontinuous at the cutoff type that is indifferent between participating and not participating. In Appendix B, I also demonstrate how to apply the method to delegation problems without participation constraints (Amador and Bagwell, 2013).

Literature Review. The vast literature on monopoly regulation dates back to Baron and Myerson (1982), who study the problem with transfers (see also Lewis and Sappington (1988); Laffont and Tirole (1986)). More recently, Amador and Bagwell (2022) study the problem without transfers and provide sufficient conditions under which the price-cap regulation is optimal. Wei and Zou (2025) study monopolist regulation with subsidies (i.e., one-sided transfers) using the optimal control method and provide conditions under which the optimal policy is laissez-faire or a progressive price cap.¹

Because there are no transfers, the problem can be formulated as a delegation problem with participation constraints. A wide range of methods have been used to solve delegation problems (with or without participation constraints). Amador and Bagwell (2013, 2022) develop the cumulative Lagrangian method. A contemporaneous work by Amador et al. (2026) generalizes the Lagrangian approach to solve delegation problems without participation constraints, while allowing for jumps in the optimal allocation. Kleiner et al. (2021) use the majorization method for linear delegation. Kolotilin and Zapechelnyuk (2025) establish the equivalence to Bayesian persuasion and use techniques in the persuasion literature. Saran (2025) uses a dynamic optimization approach to delegation problems.

The maximum principles in the existing optimal control literature (see, e.g., Léonard and Long (1992); Seierstad and Sydsæter (1987)) are *almost* applicable to delegation problems, but two technical issues need to be addressed for mathematical rigor. The first issue is to handle the monotonicity constraint on the allocation, especially given that the optimal allocation in delegation can be discontinuous (due to the lack of transfers). The standard approach (e.g., Guesnerie and Laffont, 1984) is to use the derivative of the

¹Because they allow one-sided transfers, maximum principles and sufficiency theorems involving state *inequality* constraints in the existing literature are applicable.

allocation as a control variable and impose the monotonicity constraint by requiring it to be nonnegative, which does not work when the allocation is discontinuous because its derivative does not exist. Hellwig (2008) formulates a maximum principle that provides necessary conditions for optimality, which handles the monotonicity constraint without assuming continuity of the allocation. Heuristically, his method still treats the derivative of the allocation as a control variable but defines its value to be $+\infty$ when the allocation jumps upward.²

The second challenge arises from the equality constraint on the state variable when establishing the *sufficiency* of the maximum principle. Because there are no transfers in delegation problems, the envelope equation is a *state equality constraint* involving state variables only. In the presence of state constraints, existing sufficiency theorems in the optimal control theory (e.g., Léonard and Long (1992, Theorem 10.3.2); Seierstad and Sydsæter (1987, Chapter 6.2)) only consider state *inequality* constraints that are quasiconcave in state variables. While it seems plausible to replace the equality constraint with two inequality constraints and apply their sufficiency theorems, this usually does not work because one of the inequalities will not be quasiconcave. Another approach is to differentiate the constraint, which does not work either because the allocation may be discontinuous. Therefore, when transfers are unavailable, existing sufficiency theorems cannot be applied to show the sufficiency of the maximum principles.^{3,4}

Outside delegation and the optimal control theory, Toikka (2011) extends the Myersonian ironing technique without assuming continuity of the allocation. Transfer is also crucial in this method because it ensures incentive compatibility of the solution to the problem after substituting the envelope condition into the objective function.

²Jumps in the state variable can be handled similarly using Theorems 7 and 8 in Seierstad and Sydsæter (1987, pp. 196–199).

³In Hellwig (2008, 2010), the transfer serves as a control variable and appears in the equality constraint, so existing sufficiency theorems for problems with mixed constraints (containing both state and control variables) (Léonard and Long, 1992, Theorem 6.3.2) can be applied.

⁴One can also incorporate the equality constraint into the objective function and then relax the equality constraint to a (concave) inequality constraint (à la Kartik et al. (2021), who use Lagrangian instead of the optimal control method). Then, existing sufficiency theorems for state inequality constraints are applicable. See Remark 7 for details.

2 The Regulator's Problem

2.1 Setup

I follow the model setup by [Amador and Bagwell \(2022\)](#). Consider a monopolist (i.e., agent) who faces an inverse demand function given by $P(q)$, where $q \in Q$ is the quantity produced. We assume the monopolist faces a constant marginal cost of production $\gamma \in \Gamma = [\underline{\gamma}, \bar{\gamma}]$ (where $\bar{\gamma} > \underline{\gamma} > 0$), which is his private information and is drawn from a commonly known distribution function $F(\gamma)$ with a continuous density $f(\gamma) > 0$. The monopolist also faces a fixed cost $\sigma \geq 0$, which is common knowledge. The regulator (i.e., principal) aims to maximize a weighted social welfare function in which profits receive weight $\alpha \in (0, 1]$.

Define

$$\begin{aligned} b(q) &\equiv P(q)q \\ v(q) &\equiv \int_0^q P(z)dz - P(q)q \\ w(\gamma, q) &\equiv -\gamma q + b(q) + \frac{1}{\alpha}v(q) \end{aligned}$$

Assume $P'(q) < 0 < P(q)$ and that $b(q)$ is strictly concave so that (1) $v'(q) = -P'(q)q > 0$ and (2) $w_{qq}(\gamma, q) \leq 0$ for all $q \in Q$ and $\gamma \in \Gamma$.

Define the principal-optimal quantity as

$$q_e(\gamma) = \arg \max_{q \in Q} w(\gamma, q).$$

Define the profit function (gross of the fixed cost) as

$$\pi(\gamma, q) \equiv -\gamma q + b(q).$$

Define the agent's flexible quantity (under a laissez-faire policy) as

$$q_f(\gamma) = \arg \max_{q \in Q} \pi(\gamma, q).$$

Because $w_q(\gamma, q_f(\gamma)) = \frac{1}{\alpha}v'(q_f(\gamma)) > 0$, we have $q_e(\gamma) > q_f(\gamma)$ for all $\gamma \in \Gamma$. In other words, the agent has an incentive to produce less than the principal's optimal quantity level (i.e., downward bias), and the principal wants to incentivize the agent to produce more.

Define

$$q_i(\gamma) = \{q > q_f(\gamma) : -\gamma q + b(q) = \sigma\}$$

as the quantity level at which an agent is indifferent between producing or not, which depends on the fixed cost $\sigma \geq 0$. Assume $\pi(\gamma, q_f(\gamma)) > \sigma$ for all $\gamma \in \Gamma$, so $q_i(\gamma) > q_f(\gamma)$ is well-defined and $q_i(\gamma)$ is decreasing for all $\gamma \in \Gamma$. Moreover, for all $\gamma \in \Gamma$, we have

$$w_q(\gamma, q_i(\gamma)) = \frac{\sigma}{q_i(\gamma)} - \left(\frac{1}{\alpha} - 1\right) q_i(\gamma) P'(q_i(\gamma)) \geq 0.$$

Thus, $q_e(\gamma) \geq q_i(\gamma)$. In other words, the agent's downward bias is so large that the principal would always want to push the agent to produce more until the agent's participation constraint binds.

Throughout the paper, “increasing” (“decreasing”) means “nondecreasing” (“nonincreasing”) unless otherwise specified.

2.2 Truncated problem

Given the cutoff type γ_t , Consider the following truncated problem [P_t]

$$\max_{q_t: \Gamma_t(\gamma_t) \rightarrow \mathcal{Q}} \int_{\Gamma_t(\gamma_t)} (w(\gamma, q_t(\gamma)) - \sigma) dF(\gamma) \quad (1)$$

$$\text{subject to} \quad -\gamma q_t(\gamma) + b(q_t(\gamma)) - \sigma - \int_{\gamma}^{\gamma_t} q_t(\tilde{\gamma}) d\tilde{\gamma} = \bar{U}, \quad \text{for all } \gamma \in \Gamma_t(\gamma_t) \quad (2)$$

$$q_t(\gamma) \text{ decreasing,} \quad \text{for all } \gamma \in \Gamma_t(\gamma_t) \quad (3)$$

$$-\gamma q_t(\gamma) + b(q_t(\gamma)) - \sigma \geq 0, \quad \text{for all } \gamma \in \Gamma_t(\gamma_t), \quad (4)$$

where $\bar{U} \equiv -\gamma_t q_t(\gamma_t) + b(q_t(\gamma_t)) - \sigma$.

Now we look for conditions for the following price cap (quantity floor) allocation to be optimal in the truncated problem:

$$q_t^*(\gamma | \gamma_t) = \begin{cases} q_f(\gamma); & \gamma \in [\underline{\gamma}, \gamma_H(\gamma_t)) \\ q_i(\gamma_t); & \gamma \in [\gamma_H(\gamma_t), \gamma_t] \end{cases} \quad (5)$$

where $\gamma_H(\gamma_t) = \max\{b'(q_i(\gamma_t)), \underline{\gamma}\}$ is the endpoint of the flexible region (where $q^*(\gamma) = q_f(\gamma)$). If $b'(q_i(\gamma_t)) \geq \underline{\gamma}$, then $\gamma_H(\gamma_t) = b'(q_i(\gamma_t))$; otherwise, $\gamma_H(\gamma_t) = \underline{\gamma}$, so the optimal allocation has no flexible region.

2.2.1 Sufficient conditions

Define $\kappa = \inf_q \{w_{qq}(q, \gamma)/b''(q)\} = 1 + \inf_q \{v''(q)/\alpha b''(q)\}$ and

$$L(\gamma|\gamma_t) = \frac{1}{\gamma_t - \gamma} \left[\int_{\gamma}^{\gamma_t} w_q(\tilde{\gamma}, q_i(\gamma_t)) f(\tilde{\gamma}) d\tilde{\gamma} + \kappa(\gamma - b'(q_i(\gamma_t)))(F(\gamma_t) - F(\gamma)) \right], \gamma \in [b'(q_i(\gamma_t)), \gamma_t]$$

and

$$A^* = L(b'(q_i(\gamma_t))|\gamma_t) = \frac{1}{\gamma_t - b'(q_i(\gamma_t))} \int_{b'(q_i(\gamma_t))}^{\gamma_t} w_q(\tilde{\gamma}, q_i(\gamma_t)) f(\tilde{\gamma}) d\tilde{\gamma} \geq 0, \quad (6)$$

which is weakly smaller than AB's constant A . Note that $A^* = A^*(\gamma_t)$ is a function of the cutoff type γ_t and also depends on fixed cost σ (through $q_i(\gamma_t)$).

Proposition 1. *The price-cap allocation solves the regulator's truncated problem [Pt] if the following two conditions hold:*

Condition (i*). $L(\gamma|\gamma_t) \geq A^* \equiv L(b'(q_i(\gamma_t))|\gamma_t)$ for all $\gamma \in [\gamma_H(\gamma_t), \gamma_t]$.

Condition (ii). $\kappa F(\gamma) + w_q(\gamma, q_f(\gamma)) f(\gamma)$ is increasing in γ for $\gamma \in [\underline{\gamma}, \gamma_H(\gamma_t)]$.

2.2.2 Comparison with AB

The sufficient conditions that are weaker than AB's (for the truncated problem), as they do not require the existence of a flexible pricing region below the price cap (i.e., $\gamma_H(\gamma_t) > \underline{\gamma}$). Therefore, they accommodate cases in which the optimal price-cap regulation induces a bang-bang allocation, where the firm either shuts down or always sets the price at the cap.

To see this, AB's Proposition 1 shows that the price-cap allocation is optimal under condition (ii) and the following condition (i):

Condition (i). $G(\gamma|\gamma_t) \leq G(\gamma_t|\gamma_t)$ for all $\gamma \in [\gamma_H(\gamma_t), \gamma_t]$, where

$$G(\gamma|\gamma_t) \equiv -\kappa F(\gamma_t) + \kappa \left[\frac{\gamma - b'(q_i(\gamma_t))}{\gamma - \gamma_H(\gamma_t)} \right] F(\gamma) + \frac{1}{\gamma - \gamma_H(\gamma_t)} \int_{\gamma_H(\gamma_t)}^{\gamma} w_q(\tilde{\gamma}, q_i(\gamma_t)) f(\tilde{\gamma}) d\tilde{\gamma}$$

As shown in the following observation, condition (i*) is less restrictive than AB's condition (i): they are equivalent if there exists a nondegenerate flexible pricing region—i.e., $b'(q_i(\gamma_t)) \geq \underline{\gamma}$. However, condition (i*) is strictly *weaker* if the flexible pricing region is degenerate—i.e., $b'(q_i(\gamma_t)) < \underline{\gamma}$.

Observation 1. *AB's condition (i) implies condition (i*); the converse is true if and only if $b'(q_i(\gamma_t)) \geq \underline{\gamma}$.*

Proof. AB's condition (i) is equivalent to $L(\gamma|\gamma_t) \geq L(\gamma_H(\gamma_t)|\gamma_t)$ for all $\gamma \in [\gamma_H(\gamma_t), \gamma_t)$. Thus, condition (i*) is weaker because $A^* \equiv L(b'(q_i(\gamma_t))|\gamma_t) \leq L(\gamma_H(\gamma_t)|\gamma_t)$ with equality if and only if $b'(q_i(\gamma_t)) \geq \underline{\gamma}$. \square

Technically, the multiplier proposed by AB is $A \equiv G(\gamma_t|\gamma_t) = L(\gamma_H(\gamma_t)|\gamma_t) \geq L(b'(q_i(\gamma_t))|\gamma_t) \equiv A^*$. Thus, the multiplier A^* here is more precise.

Economic Relevance. The weaker conditions do not require a flexible pricing region and thus accommodate cases where the optimal price-cap regulation induces a bang-bang allocation—the firm either shuts down or sets the price at the cap. This bang-bang allocation may arise when the price cap is low or when the optimal regulation takes the form of a take-it-or-leave-it offer (or fixed-price regulation) where the firm either accepts the price or shuts down. The bang-bang allocation can be more realistic because, in practice, the monopolist in the market rarely sets the price below the cap—in other words, price caps are often binding.

Graphical Illustration. On the bunching region $[b'(q_i(\gamma_t)), \gamma_t)$, define

$$s(\gamma) = w_q(\gamma, q_i(\gamma_t))f(\gamma) + \kappa f(\gamma)(\gamma - b'(q_i(\gamma_t))) + \kappa(F(\gamma) - F(\gamma_t)) \quad (7)$$

and $S(\gamma) = \int_{\underline{\gamma}}^{\gamma} s(x) dx$. Therefore, we have

$$L(\gamma|\gamma_t) = \frac{1}{\gamma_t - \gamma} \left[\int_{\gamma}^{\gamma_t} w_q(\tilde{\gamma}, q_i(\gamma_t))f(\tilde{\gamma}) d\tilde{\gamma} + \kappa(\gamma - b'(q_i(\gamma_t)))(F(\gamma_t) - F(\gamma)) \right] = \frac{S(\gamma) - S(\gamma_t)}{\gamma - \gamma_t}$$

is the slope of the line connecting $(\gamma, S(\gamma))$ and $(\gamma_t, S(\gamma_t))$. In particular, $A^* = L(b'(q_i(\gamma_t))|\gamma_t)$ is the slope of the secant line connecting $(b'(q_i(\gamma_t)), S(b'(q_i(\gamma_t))))$ and $(\gamma_t, S(\gamma_t))$.

For illustration purposes, consider $w(\gamma, q) = \frac{1}{2}b(q) - (\gamma - \frac{1}{2})q$ so that $\kappa = \frac{1}{2}$ and

$$s(\gamma) = \frac{1}{2}(1 - \gamma)f(\gamma) + \frac{1}{2}(F(\gamma) - F(\gamma_t)).$$

The secant line with a slope of A^* is illustrated by the red dashed lines in Figure 1.

By contrast, AB's function $G(\gamma|\gamma_t)$ is the slope of the line connecting $(\gamma, S(\gamma))$ and $(\gamma_H(\gamma_t), S(\gamma_H(\gamma_t)))$. In particular, their multiplier $A = G(\gamma_t|\gamma_t)$ is the slope of the line connecting $(\gamma_H(\gamma_t), S(\gamma_H(\gamma_t)))$ and $(\gamma_t, S(\gamma_t))$, as illustrated by the purple dashed line in the right panel of Figure 1 (and the red dashed lines in the left panel).

Figure 1 illustrates the comparison between condition (i*) and AB's condition (i). In the left panel, the red dashed line represents both $A^* = L(b'(q_i(\gamma_t))|\gamma_t)$ and AB's

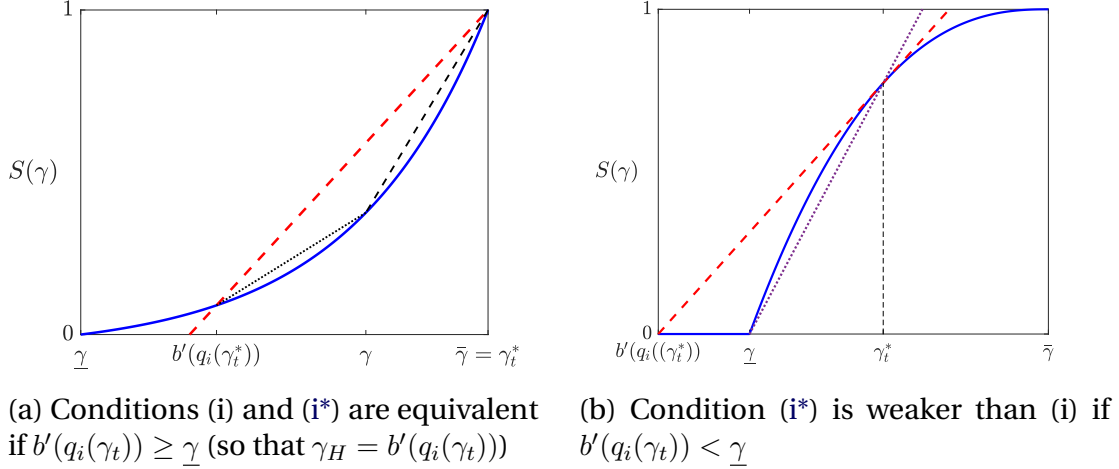


Figure 1: Condition (i*) vs. AB's Condition (i)

$A = G(\gamma_t|\gamma_t) = L(\gamma_H(\gamma_t)|\gamma_t)$, which coincide because $b'(q_i(\gamma_t)) \geq \underline{\gamma}$ (and hence $b'(q_i(\gamma_t)) = \gamma_H(\gamma_t)$). For a fixed $\gamma \in [\gamma_t, \gamma_H(\gamma_t)]$, the black dashed line represents $L(\gamma|\gamma_t)$, while the black dotted line represents $G(\gamma|\gamma_t)$ in AB; the former has a higher slope than the red dashed line (which has a slope of A^*) if and only if the latter has a lower slope than the red line. Thus, AB's Condition (i) and condition (i*) are equivalent if $b'(q_i(\gamma_t)) \geq \underline{\gamma}$.

In the right panel, the purple dashed line represents AB's $A = L(\gamma_H(\gamma_t)|\gamma_t)$, while the red dashed line represents $A^* = L(b'(q_i(\gamma_t))|\gamma_t)$. Unlike the left panel, because $b'(q_i(\gamma_t)) < \underline{\gamma}$, the function S (or L) satisfies condition (i*) but violates AB's condition (i). Thus, condition (i*) is strictly weaker than AB's condition (i) if $b'(q_i(\gamma_t)) < \underline{\gamma}$.

2.3 Untruncated problem

The untruncated problem [P] is given by

$$\max_{q:\Gamma \rightarrow Q} \int_{\Gamma} (w(\gamma, q(\gamma)) - \sigma \mathbf{1}(q(\gamma))) dF(\gamma) \quad (8)$$

$$\text{subject to} \quad -\gamma q(\gamma) + b(q(\gamma)) - \sigma \mathbf{1}(q(\gamma)) - \int_{\gamma}^{\bar{\gamma}} q(\tilde{\gamma}) d\tilde{\gamma} = \bar{U}, \quad \text{for all } \gamma \in \Gamma \quad (9)$$

$$q(\gamma) \text{ decreasing, for all } \gamma \in \Gamma \quad (10)$$

$$-\gamma q(\gamma) + b(q(\gamma)) - \sigma \mathbf{1}(q(\gamma)) \geq 0, \quad \text{for all } \gamma \in \Gamma, \quad (11)$$

where $\bar{U} \equiv -\bar{\gamma}q(\bar{\gamma}) + b(q(\bar{\gamma})) - \sigma \mathbf{1}(q(\bar{\gamma}))$.

We look for conditions for the following price cap (quantity floor) allocation to be

optimal in the untruncated problem,

$$q^*(\gamma) = \begin{cases} q_f(\gamma); & \gamma \in [\underline{\gamma}, \gamma_H(\gamma_t)) \\ q_i(\gamma_t); & \gamma \in [\gamma_H(\gamma_t), \gamma_t] \\ 0; & \gamma \in (\gamma_t, \bar{\gamma}] \end{cases} \quad (12)$$

I first solve the problem without fixed cost (i.e., $\sigma = 0$) and then incorporate the fixed cost later.

2.3.1 Sufficient Conditions (without Fixed Cost)

Define

$$L(\gamma|\gamma_t) = \begin{cases} \frac{1}{\gamma - \gamma_t} \left[\int_{\gamma_t}^{\gamma} w_q(\tilde{\gamma}, 0) f(\tilde{\gamma}) d\tilde{\gamma} + \kappa(\gamma - b'(0))(F(\gamma) - F(\gamma_t)) \right], & \text{if } \gamma \in (\gamma_t, \bar{\gamma}] \\ \frac{1}{\gamma_t - \gamma} \left[\int_{\gamma}^{\gamma_t} w_q(\tilde{\gamma}, q_i(\gamma_t)) f(\tilde{\gamma}) d\tilde{\gamma} + \kappa(\gamma - b'(q_i(\gamma_t)))(F(\gamma_t) - F(\gamma)) \right], & \text{if } \gamma \in [b'(q_i(\gamma_t)), \gamma_t]. \end{cases} \quad (13)$$

Recall that $A^* = L(b'(q_i(\gamma_t))|\gamma_t)$.

Proposition 2. *The price cap allocation with cutoff type $\gamma_t \in [\underline{\gamma}, \bar{\gamma}]$ is optimal if the following three conditions hold:*

Condition (i*). $L(\gamma|\gamma_t) \geq L(b'(q_i(\gamma_t))|\gamma_t) = A^*$ for all $\gamma \in [\gamma_H(\gamma_t), \gamma_t)$.

Condition (i'). $L(\gamma|\gamma_t) \leq L(b'(q_i(\gamma_t))|\gamma_t) = A^*$ for all $\gamma \in (\gamma_t, \bar{\gamma}]$.

Condition (ii). $\kappa F(\gamma) + w_q(\gamma, q_f(\gamma)) f(\gamma)$ is increasing in γ for $\gamma \in [\underline{\gamma}, \gamma_H(\gamma_t))$.

Condition (i*) replaces AB's condition (i), as discussed in the previous subsection. Condition (i') is a new condition that ensures $q = 0$ is optimal in the exclusion region.

Define

$$s(\gamma) = \begin{cases} w_q(\gamma, 0) f(\gamma) + \kappa f(\gamma)(\gamma - b'(0)) + \kappa(F(\gamma) - F(\gamma_t)), & \text{if } \gamma \in (\gamma_t, \bar{\gamma}] \\ w_q(\gamma, q_i(\gamma_t)) f(\gamma) + \kappa f(\gamma)(\gamma - b'(q_i(\gamma_t))) + \kappa(F(\gamma) - F(\gamma_t)), & \text{if } \gamma \in [b'(q_i(\gamma_t)), \gamma_t) \\ w_q(\gamma, q_f(\gamma)) f(\gamma) + \kappa(F(\gamma) - F(\gamma_t)), & \text{if } \gamma \in [\underline{\gamma}, b'(q_i(\gamma_t))) \end{cases} \quad (14)$$

and $S(\gamma) = \int_{\underline{\gamma}}^{\gamma} s(x) dx$ so that $L(\gamma|\gamma_t) = \frac{S(\gamma_t) - S(\gamma)}{\gamma_t - \gamma}$. Then, conditions (i*) and (i') can be combined into a single condition that involves $s(\gamma)$:

Condition (S). $\int_{\gamma_t}^{\gamma} s(\gamma) d\gamma \leq (\gamma - \gamma_t)A^*$ for all $\gamma \in [\gamma_H(\gamma_t), \bar{\gamma}]$.⁵

Graphically, the conditions mean the line ℓ connecting γ_t and $b'(q_i(\gamma_t))$ on $S(\gamma)$ (red dashed line in Figure 2), which has a slope of $L(b'(q_i(\gamma_t))|\gamma_t) = A^*$, lies below S for all $\gamma \in [\gamma_H(\gamma_t), \bar{\gamma}]$. In other words, ℓ is the supporting hyperplane (line) of the hypograph of $S(\gamma)$ on $\gamma \in [\gamma_H(\gamma_t), \bar{\gamma}]$ containing γ_t . In particular, if $\gamma_t < \bar{\gamma}$, then ℓ must be tangent to $S(\gamma)$ at γ_t .

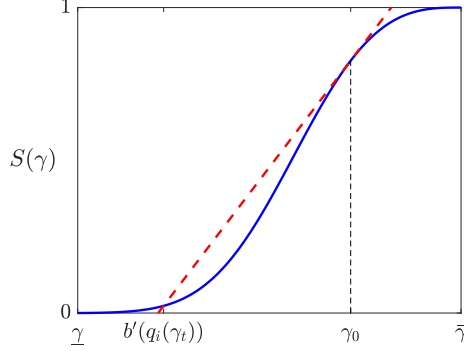


Figure 2: Graphic Illustration of Conditions (i*), (ii) and (i')

Condition (ii) in AB is still required to guarantee concavity in the flexible region. Condition (ii) implies that $S(\gamma)$ is convex (i.e., $s(\gamma)$ is increasing) on $[\underline{\gamma}, b'(q_i(\gamma_t))]$.

The conditions in the proposition weakens AB's sufficient conditions in two ways. First, because of the more precise multiplier, I replace AB's condition (i) by a weaker condition that does not hinge on the existence of the flexible region (i.e., $b'(q_i(\gamma_t)) \geq \underline{\gamma}$). Second, the sufficient conditions need only hold at the endogenously determined cutoff γ_t . Consequently, the price-cap regulation that induces a bang-bang allocation where the firm either shuts down or sets the price at the price cap can also be optimal. This bang-bang allocation can also be implemented by a take-it-or-leave-it offer (or fixed-price regulation), where the firm either accepts the fixed price set by the regulator or shuts down, as shown in the following corollary.

Corollary 2.1. *If conditions (i*) and (i') hold at some $\gamma_t \in [\underline{\gamma}, \bar{\gamma}]$ such that $b'(q_i(\gamma_t)) \leq \underline{\gamma}$, a bang-bang allocation where the firm either shuts down or sets the price at the price cap is optimal.*

⁵If $\gamma_t < \bar{\gamma}$, this condition implies:

$$\int_{\gamma_t}^{\gamma} s(\tilde{\gamma}) d\tilde{\gamma} \leq s(\gamma_t)(\gamma - \gamma_t) \text{ for all } \gamma \in [\gamma_H(\gamma_t), \bar{\gamma}], \text{ with equality at } \gamma = b'(q_i(\gamma_t)).$$

Proof. This follows from Proposition 2 by noting that condition (ii) holds vacuously when $b'(q_i(\gamma_t)) \leq \underline{\gamma}$ (i.e., the optimal allocation has no flexible region). \square

The following corollary provides easy-to-check sufficient conditions under which the price-cap regulation (including one that induces a bang-bang allocation) is optimal in terms of the shape of $s(\gamma)$.

Corollary 2.2. *The price cap allocation is optimal if $s(\gamma)$ is unimodal. In particular,*

- *If $s(\gamma)$ is increasing, the price cap allocation without exclusion is optimal;*
- *If $s(\gamma)$ is decreasing, a bang-bang allocation (or take-it-or-leave-it offer) where the firm either shuts down or sets the price at the price cap is optimal.*

Proof Sketch. If $s(\gamma)$ is unimodal, then $S(\gamma)$ is convex-concave, then the three conditions in Proposition 2 hold at some $\gamma_t \in [\underline{\gamma}, \bar{\gamma}]$, which is evident from Figure 2. In particular, if $s(\gamma)$ is increasing, then $S(\gamma)$ is convex, then conditions (i*) and (i') hold at some $\gamma_t = \bar{\gamma}$, so the optimal allocation has no exclusion. If $s(\gamma)$ is decreasing, then $S(\gamma)$ is concave, then conditions (i*) and (i') hold at some $\gamma_t \in [\underline{\gamma}, \bar{\gamma}]$ such that $b'(q_i(\gamma_t)) \leq \underline{\gamma}$, so the optimal allocation has no flexible region. \square

Remark 1. As a normalization, define $\tilde{s}(\gamma) = s(\gamma) + \kappa F(\gamma_t)$, which differs from s by a constant:

$$\tilde{s}(\gamma) = \begin{cases} w_q(\gamma, 0)f(\gamma) + \kappa f(\gamma)(\gamma - b'(0)) + \kappa F(\gamma), & \text{if } \gamma \in (\gamma_t, \bar{\gamma}] \\ w_q(\gamma, q_i(\gamma_t))f(\gamma) + \kappa f(\gamma)(\gamma - b'(q_i(\gamma_t))) + \kappa F(\gamma), & \text{if } \gamma \in [b'(q_i(\gamma_t)), \gamma_t) \\ w_q(\gamma, q_f(\gamma))f(\gamma) + \kappa F(\gamma), & \text{if } \gamma \in [\underline{\gamma}, b'(q_i(\gamma_t))) \end{cases} \quad (15)$$

which satisfies $\tilde{s}(\gamma) = 0$ for all $\gamma < \underline{\gamma}$ and $\tilde{s} = \kappa F(\gamma)$ for all $\gamma > \bar{\gamma}$. Because s is increasing (decreasing) if and only if \tilde{s} is increasing (decreasing), Corollary 2.2 holds with $\tilde{s}(\gamma)$ in place of $s(\gamma)$.

2.3.2 Comparison to Amador and Bagwell (2022)

The conditions I propose are weaker than those in AB in two ways. First, as in the truncated problem, the multiplier I propose is more precise, so condition (i*) is weaker than their condition (i), particularly when $b'(q_i(\gamma_t)) < \underline{\gamma}$ (see Figure 1), and thus does not require a flexible pricing region below the price cap

Second, unlike AB, conditions (i*) and (ii) need not hold for all $\gamma_t \in (\underline{\gamma}, \bar{\gamma}]$. Instead, they need only hold for the endogenously determined cutoff γ_t determined by conditions (i*) and (i'). Thus, exclusion can be optimal even when there is no fixed cost.

Consequently, this approach yields weaker sufficient conditions under which exclusion can be optimal even when there is no fixed cost. Furthermore, in contrast to AB, a bang-bang allocation where the firm either shuts down or always sets the price at the price cap can also be optimal (e.g., when $s(\gamma)$ is decreasing).

It is worth mentioning that AB considers a fixed cost $\sigma \geq 0$ in their analysis, while the results so far assume $\sigma = 0$. In the next subsection, I will incorporate a fixed cost σ into the analysis and show that the sufficient conditions in Corollary 2.2 still hold.

2.4 Incorporating the Fixed Cost

Now I incorporate a fixed cost $\sigma \geq 0$ to the analysis and show that the sufficient conditions in Corollary 2.2 still hold. Consistent with AB (Assumption 2), I assume $\sigma \leq \pi(\bar{\gamma}, q_f(\bar{\gamma}))$.

Note that the fixed cost σ affects the optimization problem through the agent's participation constraint and the principal's objective function. First, it affects the agent's participation constraint and thus decreases $q_i(\gamma) = \{q > q_f(\gamma) : -\gamma q + b(q) = \sigma\}$. Consequently, it decreases the multiplier A^* given by

$$A^* = \frac{1}{\gamma_t - b'(q_i(\gamma_t))} \int_{\gamma_H(\gamma_t)}^{\gamma_t} w_q(\gamma, q_i(\gamma_t)) dF(\gamma),$$

Consider an auxiliary problem where the fixed cost σ is incorporated into the participation constraint only and thus the definitions of $q_i(\gamma)$ and A^* (but not the objective function). Then, the previous analysis still goes through and implies an optimal cutoff type, denoted by $\tilde{\gamma}_t$.

Second, the fixed cost σ affects the principal's objective function and leads to more exclusion. Denote by $\gamma_t(\sigma)$ the optimal cutoff type when the fixed cost σ is *fully* incorporated (i.e., into both the participation constraint and the objective function). Compared to the auxiliary problem where it is only incorporated into the participation constraint, the fixed cost σ further reduces the principal's payoff, it must be $\gamma_t(\sigma) \leq \tilde{\gamma}_t$. Thus, $\tilde{\gamma}_t$ provides an upper bound for the optimal cutoff type where there is a fixed cost σ . Hence, going back to the truncated problem $[P_t]$, it suffices to check whether the sufficient conditions are satisfied for all $\gamma_t \leq \tilde{\gamma}_t$, which is true if $s(\gamma)$ is unimodal.

Proposition 3. *Assume the fixed cost $\sigma \in [0, \pi(\bar{\gamma}, q_f(\bar{\gamma}))]$. Then, the price-cap allocation is optimal if $s(\gamma)$ is unimodal. In particular, if $s(\gamma)$ is decreasing, a bang-bang allocation (or*

take-it-or-leave-it offer) is optimal.

Proof Sketch. If $s(\gamma)$ is unimodal, then S is convex-concave, and conditions (i*) and (ii) hold at some $\tilde{\gamma}_t \in [\underline{\gamma}, \bar{\gamma}]$; moreover, they also hold for all $\gamma_t \leq \tilde{\gamma}_t$, which is evident from Figure 2. \square

3 Constant Curvature of Demand

In this section, I apply my result to a family of (inverse) demand functions $P(q)$ with constant *curvature of demand* (see, e.g., Seade, 1980; Aguirre et al., 2010; Mrázová and Neary, 2017)

$$\rho(q) \equiv -\frac{P''(q)q}{P'(q)} = \rho \text{ (constant)}.$$

Assume $\rho < 2$, so the revenue $b(q) = P(q)q$ is strictly concave. This family of demand functions must take the form of

$$P(q) = \begin{cases} P_1 - \beta \ln q, & \text{if } \rho = 1 \\ P_1 - \frac{\beta}{1-\rho}(q^{1-\rho} - 1) \equiv P_0 - \frac{\beta}{1-\rho}q^{1-\rho}, & \text{if } \rho < 2 \text{ and } \rho \neq 1 \end{cases}$$

where $P_1, \beta > 0$ are constants and $P_0 = P_1 + \frac{\beta}{1-\rho}$.

For this family of demand functions, the elasticity of demand is given by

$$\varepsilon(q) \equiv -\frac{P(q)}{qP'(q)} = -\frac{1}{1-\rho} + \frac{P_0}{\beta}q^{\rho-1}$$

For $\rho \neq 1$, denote the constant term by $\epsilon = -\frac{1}{1-\rho} \in (-\infty, 0) \cup (1, \infty)$ (because $\rho < 2$), which is the limit of the elasticity $\varepsilon(q)$ as $q \rightarrow 0$ ($q \rightarrow \infty$) if $\rho > 1$ ($\rho < 1$). In particular, if $P_0 = 0$ and $\rho > 1$, then $\epsilon = -\frac{1}{1-\rho} > 1$ is the elasticity of demand.

Furthermore, for this family of demand, we have $b(q) \equiv P(q)q$, $v(q) \equiv \int_0^q P(z)dz - P(q)q$, and thus

$$w(\gamma, q) \equiv -\gamma q + b(q) + \frac{1}{\alpha}v(q) = c \cdot b(q) - d(\gamma)q,$$

where $\alpha \in (0, 1]$ is the welfare weight on the firm's profit,

$$c = 1 - \frac{1}{\alpha} \frac{1-\rho}{2-\rho} = 1 - \frac{1}{\alpha} \frac{1}{1-\epsilon}, \text{ and } d(\gamma) = \gamma + (c-1)P_0.^6$$

⁶If $\rho = 1$, then $c-1 = -\frac{1}{\alpha} \frac{1-\rho}{2-\rho} = 0$ and $P_0 = P_1 + \frac{\beta}{1-\rho} \rightarrow \infty$, so we have $(c-1)P_0 = -\frac{\beta}{\alpha}$ and thus $d(\gamma) = \gamma - \beta/\alpha$.

Given the functional form of $w(\gamma, q)$, Proposition 3 implies the following.

Lemma 1 (Constant curvature demand). *Assume that the demand function has a constant curvature $\rho < 2$. The price-cap allocation is optimal if*

$$\tilde{s}(\gamma) = (c\gamma - d(\gamma))f(\gamma) + cF(\gamma)$$

is unimodal. In particular,

- *if $\tilde{s}(\gamma)$ is increasing, the optimal price-cap allocation involves no exclusion when the fixed cost is zero;*
- *if $\tilde{s}(\gamma)$ is decreasing, a bang-bang allocation (or take-it-or-leave-it offer) is optimal.*

Proof. Substituting the envelope equation $b(q(\gamma)) = \int_{\underline{\gamma}}^{\bar{\gamma}} q(\tilde{\gamma}) d\tilde{\gamma} + \gamma q(\gamma) + (\text{constant})$ into the principal's objective function and using integration by parts, we have

$$\begin{aligned} \int_{\underline{\gamma}}^{\bar{\gamma}} w(\gamma, q(\gamma))f(\gamma) d\gamma &= \int_{\underline{\gamma}}^{\bar{\gamma}} ((c\gamma - d(\gamma))f(\gamma) + cF(\gamma)) q(\gamma) d\gamma + (\text{constant}) \\ &= \int_{\underline{\gamma}}^{\bar{\gamma}} q(\gamma)\tilde{s}(\gamma) d\gamma + (\text{constant}) \end{aligned}$$

which is linear in q for all γ . Thus, the problem is equivalent to a problem with one where the principal's payoff is $w(q, \gamma) = q$ (so that $w_q(q, \gamma) = 1$ and $\kappa = 0$) and the density is $\tilde{s}(\gamma)$. Proposition 2 (and hence Proposition 3) applies directly. \square

Remark 2. It is unnecessary to assume that $w(\gamma, q) = c \cdot b(q) - d(\gamma)q$ is concave in q (i.e., $c \geq 0$).

Logarithmic demand ($\rho = 1$). First, I consider logarithmic demand $P(q) = P_1 - \beta \ln q$, which has a constant curvature $\rho = 1$, and show that price-cap is optimal if f is log-concave.

Corollary 3.1 (Logarithmic demand). *Assume $P(q) = P_1 - \beta \ln q$ (i.e., $\rho = 1$), where $P_1, \beta > 0$. Then, $w(\gamma, q) = b(q) - (\gamma - \beta/\alpha)q$, and price-cap allocation is optimal if one of the following holds:*

1. $f(\gamma)$ is log-concave;
2. $\frac{f'(\gamma)}{f(\gamma)} \geq -\frac{\alpha}{\beta}$ (which holds if $f(\gamma)$ is increasing);
3. $\frac{f'(\gamma)}{f(\gamma)} \leq -\frac{\alpha}{\beta}$ (which holds if $f(\gamma)$ is strictly decreasing and α/β is sufficiently small).

Under condition 2, the optimal price-cap allocation involves no exclusion when the fixed cost is zero.

Under condition 3, the optimal allocation is bang-bang, which can also be implemented by a take-it-or-leave-it offer.

Remark 3. AB also consider logarithmic demand, but they only provide condition 2 as the sufficient condition. Similar remarks apply to linear demand and constant elasticity demand.

Log-concave demand ($\rho < 1$). Next, I consider the case where the curvature $\rho < 1$ so that $\epsilon = -\frac{1}{1-\rho} < 0$. In this case, the inverse demand function is $P(q) = P_0 - \frac{\beta}{1-\rho}q^{1-\rho}$, and the direct demand function $D(p) \equiv P^{-1}(p)$ is (strictly) log-concave in p (see, e.g., Kang and Vasserman (2022); Zou (2025)). A special case is linear demand ($\rho = 0$).

Corollary 3.2 (Log-concave demand). Assume $\rho < 1$ (i.e., $\epsilon < 0$) and $\bar{\gamma} \leq P_0$. Then, price-cap allocation is optimal if one of the following holds:

1. $f(\gamma)$ is log-concave and $\epsilon \geq 1 - \frac{2}{\alpha}$ (i.e., $\rho \leq \frac{2(1-2\alpha)}{2-\alpha}$);
2. $\frac{f'(\gamma)}{f(\gamma)} \geq \frac{2-(1-\epsilon)\alpha}{P_0-\gamma}$ (which holds if $f(\gamma)$ is increasing and $\epsilon \leq 1 - \frac{2}{\alpha}$);
3. $\frac{f'(\gamma)}{f(\gamma)} \leq \frac{2-(1-\epsilon)\alpha}{P_0-\gamma}$ (which holds if $f(\gamma)$ is decreasing and $\epsilon \geq 1 - \frac{2}{\alpha}$).

Under condition 2, the optimal price-cap allocation involves no exclusion when the fixed cost is zero.

Under condition 3, the optimal allocation is bang-bang, which can also be implemented by a take-it-or-leave-it offer.

The conditions can be further simplified for linear demand ($\rho = 0$).

Example 1 (Linear demand). Assume $P(q) = P_0 - \beta q$ (i.e., $\rho = 0$) and $\bar{\gamma} \leq P_0$. Then, the conditions reduce to

1. $f(\gamma)$ is log-concave;
2. $\frac{f'(\gamma)}{f(\gamma)} \geq \frac{2(1-\alpha)}{P_0-\gamma}$ (which holds if $f(\gamma)$ is increasing and $\alpha = 1$);
3. $\frac{f'(\gamma)}{f(\gamma)} \leq \frac{2(1-\alpha)}{P_0-\gamma}$ (which holds if $f(\gamma)$ is decreasing).

Log-convex demand ($1 < \rho < 2$). Next, I consider the case where the curvature $\rho \in (1, 2)$ so that $\epsilon = -\frac{1}{1-\rho} > 1$. In this case, the direct demand $D(p)$ is (strictly) log-convex in p . A special case is constant elasticity demand ($P_0 = 0$), where the elasticity is $\epsilon = -\frac{1}{1-\rho} > 1$.

Corollary 3.3 (Log-convex demand). *Assume $\rho \in (1, 2)$ (i.e., $\epsilon > 1$) and $\underline{\gamma} > P_0$. Then, price-cap allocation is optimal if one of the following holds:*

1. $f(\gamma)$ is log-concave;
2. $\frac{f'(\gamma)}{f(\gamma)} \geq -\frac{2+(\epsilon-1)\alpha}{\gamma-P_0}$ (which holds if $f(\gamma)$ is increasing);
3. $\frac{f'(\gamma)}{f(\gamma)} \leq -\frac{2+(\epsilon-1)\alpha}{\gamma-P_0}$.

Under condition 2, the optimal price-cap allocation involves no exclusion when the fixed cost is zero.

Under condition 3, the optimal allocation is bang-bang, which can also be implemented by a take-it-or-leave-it offer.

The conditions can be further simplified for constant elasticity demand ($P_0 = 0$).

Example 2 (Constant elasticity demand). Assume $P(q) = (\epsilon\beta)q^{-1/\epsilon}$ (i.e., $P_0 = 0$), $\epsilon > 1$, and $\underline{\gamma} \geq 0$. Then, Corollary 3.3 implies that price-cap allocation is optimal if $f(\gamma)$ is log-concave or increasing, or if $f(\gamma)$ is strictly decreasing and $\underline{\gamma}$ is sufficiently large (i.e., even the lowest-cost firm has a high marginal cost).

A special case where unimodality of f matters. Finally, I consider a special case of log-concave demand where $P_0 = 1$ and $\rho = 1 - \frac{\alpha}{2-\alpha} \in [0, 1)$ (because $\alpha \in (0, 1]$) so that $w(\gamma, q) = b(q)/2 - (\gamma - 1/2)q$. Interestingly, price caps are optimal if $f(\gamma)$ is unimodal—which holds if f is log-concave or monotone (increasing or decreasing).

Corollary 3.4. *Assume $P(q) = 1 - \frac{\beta}{1-\rho}q^{1-\rho}$, $1 - \rho = \frac{\alpha}{2-\alpha} \in (0, 1]$, and $\bar{\gamma} \leq 1$. Then, $w(\gamma, q) = b(q)/2 - (\gamma - 1/2)q$, and the price-cap allocation is optimal if $f(\gamma)$ is unimodal. In particular,*

- *if $f(\gamma)$ is increasing, the price cap allocation without exclusion is optimal when there is no fixed cost;*
- *if $f(\gamma)$ is decreasing, a bang-bang allocation (or take-it-or-leave-it offer) is optimal.*

Remark 4. Kolotilin and Zapechelnyuk (2019, Section 4.1) show that price-cap allocation is optimal if $f(\gamma)$ is unimodal when $P(q) = 1 - q$ (i.e., $\alpha = \beta = 1$).

4 A Sufficiency Theorem for Optimal Control Problems with State Equality Constraints

Consider the following optimal control problem. Assume that $f: \mathbb{R}_+ \times \mathbb{R}_+ \times [0, T] \rightarrow \mathbb{R}_+$, $v: \mathbb{R}_+ \times \mathbb{R}_+ \times [0, T] \rightarrow \mathbb{R}_+$, and $h: \mathbb{R}_+ \times [0, T] \rightarrow \mathbb{R}_+$ are continuously differentiable in all their arguments. Assume $x: [0, T] \rightarrow \mathbb{R}$ and $y: [0, T] \rightarrow \mathbb{R}$ are measurable.

$$\max_{x(t), y(t)} \int_0^T v(x(t), y(t), t) dt \quad (16)$$

$$\text{subject to } x(t) = \int_0^t f(x(\tau), y(\tau), \tau) d\tau + x(0).^7 \quad (17)$$

$$y(t) \text{ increasing} \quad (18)$$

$$x = h(y(t), t), \quad (19)$$

$$x(0), y(0) \geq 0 \quad (20)$$

$$x(T), y(T) \text{ free} \quad (21)$$

Because $x = h(y, t)$, it is without loss of generality to assume that f and v are functions of (y, t) only.⁸ Denote the Hamiltonian by

$$H = v(y, t) + p_x f(y, t) + \lambda(h(y, t) - x),$$

where p_x is the costate of x ; and λ is the Lagrangian multiplier on the equality constraint $x = h(y, t)$. While the state equation implies that x is absolutely continuous, I do not assume continuity of y , so its (Radon-Nikodym) derivative \dot{y} may not exist. If y is absolutely continuous, we can simply add its derivative \dot{y} as a control and add a term $p_y \cdot \dot{y}$ to the Hamiltonian above, where p_y is the costate of \dot{y} (see, e.g., [Guesnerie and Laffont \(1984\)](#)). Heuristically, we can use the same approach after defining $\dot{y} = +\infty$ when y jumps upward and $\dot{y} = -\infty$ if y jumps downward ([Hellwig \(2008\)](#)), which help understand the costate equation for \dot{y} . Therefore, the necessary conditions follow from the standard maximum principle after this little tweak.

⁷This constraint is equivalent to

$$\dot{x}(t) = f(x(t), y(t), t) \text{ a.e., and } x \text{ is absolutely continuous.}$$

⁸We can always redefine $\tilde{v}(y, t) = v(h(y, t), y, t)$ and $\tilde{f}(y, t) \equiv f(h(y, t), y, t)$ and, slightly abuse the notation, write $v(y, t)$ and $f(y, t)$ instead of $\tilde{v}(y, t)$ and $\tilde{f}(y, t)$.

Now that we handled discontinuity, the remaining difficulty is about sufficiency of the necessary conditions. Because $x = h(y, t)$ is a (pure) state equality constraint that does not involve any control variable, existing sufficiency theorems for mixed constraints (e.g., Léonard and Long, 1992, Theorem 6.3.2) are not applicable. Note that if the equality constraint involves a control variable (e.g., $x = h(y, w, t)$), and the problem is *easier* because the constraint becomes a mixed constraint, which makes the existing sufficiency theorems applicable (see Remark 6). The equality also makes existing sufficiency theorems for state *inequality* constraints inapplicable (e.g., Léonard and Long, 1992, Theorem 10.3.2) (see Remark 5).

Theorem 1 (Sufficient conditions for problems with state equality constraints). *Let $(x^*(t), y^*(t))$ be a feasible path. Assume there exist piecewise absolutely continuous functions $(p_x(t), p_y(t))$ and $\lambda(t)$ that satisfy the following conditions for all $t \in [0, T]$:*

1. *Costate equations:*

$$\dot{p}_x = -\frac{\partial L}{\partial x} = \lambda \quad (22)$$

$$\dot{p}_y = -\frac{\partial L}{\partial y} = -(v_y + p_x f_y + \lambda h_y) \quad (23)$$

At any point t_0 where p_x is discontinuous,

$$p_y(t_0^+) - p_y(t_0^-) = -h_y(y^*(t_0), t_0)[p_x(t_0^+) - p_x(t_0^-)]. \quad (24)$$

2. *Monotonicity of y :*

$$p_y \leq 0, \text{ and } p_y = 0 \text{ if } y^* \text{ is strictly increasing at } t \quad (25)$$

3. *Transversality conditions:*

$$p_x(0) \leq 0, p_x(0)x(0) = 0; \quad p_x(T) = 0, \quad (26)$$

$$p_y(0) \leq 0, p_y(0)y(0) = 0; \quad p_y(T) = 0 \quad (27)$$

(If $x(0), y(0)$ are free, then $p_x(0) = 0$ and $p_y(0) = 0$.)

4. *Jump conditions: If y^* has a jump discontinuity at t_0 , then*

$$p_y(t_0-) = p_y(t_0+) = 0 \quad (28)$$

$$p_x(t_0-) = p_x(t_0+) \quad (29)$$

$$H(t_0+) = H(t_0-) \quad (30)$$

5. *Concavity: The Hamiltonian $H = v + p_x f + \lambda h$ is concave in y .*

At any point where p_x is discontinuous, p_x is increasing, and h is concave in y .

Then, (x^, y^*) solves the optimization problem (16)–(21).*

Remark 5. Conditions 1–3 are the same as necessary conditions from the maximum principle formulated by Hellwig (2008, 2010). If y is absolutely continuous, then they reduce to standard necessary conditions in the optimal control theory.

Condition 4 is the necessary condition for jumps in the state variable (Seierstad and Sydsæter, 1987, Theorem 3.7). The condition $p_y(t_0) = 0$ is also implied by the monotonicity condition because y is strictly increasing at t_0 . The condition $p_x(t_0-) = p_x(t_0+)$ comes from the continuity of x .

Conditions 5 is for sufficiency. Applying sufficiency theorems in the existing optimal control theory (e.g., Léonard and Long (1992, Theorem 10.3.2) and Seierstad and Sydsæter (1987, Chapter 6.2)) would require both $x - h(y, t)$ and $h(y, t) - x$ to be quasiconcave in y , which cannot hold simultaneously unless h is monotone in y .

Remark 6. The equality constraint $x = h(y, t)$ is a pure state constraint. If it contains a control variable, e.g., $x = h(y, w, t)$, it becomes a standard mixed constraint. In this case, the necessary conditions are still Conditions 1–4, along with an additional condition $w^* \in \arg \max_w H$ for the control variable (see Hellwig (2008, 2010)). For sufficiency, following standard sufficiency theorems for problems with mixed constraints (e.g., Léonard and Long, 1992, Theorem 6.3.2), Condition 5 can be replaced by either (1) H is concave in (y, w) (Magrasián-type) or (2) $H^0 \equiv \max_w \{H : \text{s.t. } x = h(y, w, t)\}$ is concave in y (Arrow-type).

4.1 Applications to Delegation Problems

Theorem 1 can be applied to delegation problems with or without participation constraints. The previous section showcases its application to delegation problems with participation constraints. In Appendix B, I apply it to delegation problems without participation constraint (Amador and Bagwell, 2013).

5 Conclusion

In this paper, I develop a sufficiency theorem for optimal control problems with monotonicity and equality constraints on state variables, which can be applied to delegation problems with or without participation constraints.

I apply the theorem to a monopolist regulation problem without transfers (Amador and Bagwell, 2022). Thus, I provide weaker sufficient conditions for the optimality of price-cap regulation. The weaker conditions can accommodate cases where the monopolist in the market always sets the price at the cap, which is more common in practice, as monopolists in regulated markets rarely set prices below the cap.

I also focus on a family of demand functions with constant curvature, which includes linear, logarithmic, and constant elasticity demand functions. For linear demand, price caps are optimal if the cost density is log-concave or decreasing, and fixed-price regulation (which leads to a bang-bang allocation) is optimal if the density is decreasing. For log-convex demand functions with constant curvature (e.g., logarithmic and constant elasticity demand), price caps are optimal if the cost density is log-concave or increasing.

Appendix A Proofs

A.1 Proof of the Sufficiency Theorem

Proof of Theorem 1. We solve the problem using the cumulative Lagrangian method.

$$\max_{x(t), y(t)} \int_0^T v(h(y(t), t), y(t), t) dt \quad (\text{A.1})$$

$$\text{subject to } x(t) = \int_0^t f(x(\tau), y(\tau), \tau) d\tau + x(0) \quad (\text{A.2})$$

$$x(t) = h(y(t), t) \quad (\text{A.3})$$

$$y(t) \text{ increasing} \quad (\text{A.4})$$

$$x(0) \geq 0, \quad y(0) \geq 0 \quad (\text{A.5})$$

Step 1: Set up the cumulative Lagrangian. Define the (cumulative) Lagrangian as

$$\mathcal{L} = \int_0^T v dt + \int_0^T \left(h(y, t) - \int_0^t f(x(\tau), y(\tau), \tau) d\tau - x(0) \right) d\Lambda + \mu x(0)$$

where $\Lambda = \Lambda_1 - \Lambda_2$ is the differences of two increasing functions and $\mu \geq 0$.

Using integration by parts, we have

$$\mathcal{L} = \int v dt + \int \Lambda f dt + \int h d\Lambda + (\Lambda(0) + \mu)x(0) - \Lambda(T) \left(x(0) + \int_0^T f(x(\tau), y(\tau), \tau) d\tau \right)$$

Let $\Lambda = p_x$. We have the following.

Step 2: Transversality conditions. By Condition 3 (transversality), $\Lambda(T) = p_x(T) = 0$, the last term is zero. Kuhn-Tucker first-order conditions with respect to $x(0)$ is

$$\Lambda(0) + \mu \leq 0, \quad (\Lambda(0) + \mu)x(0) = 0 \quad (\text{A.6})$$

which is equivalent to the complementary-slackness condition for p_x of Condition 3.

Step 3: Concavity of Lagrangian. Now I show that \mathcal{L} is concave in y . Now the cumulative Lagrangian reduces to

$$\mathcal{L} = \int v dt + \int p_x f dt + \int h dp_x$$

By the Lebesgue decomposition theorem,

$$dp_x = \lambda dt + dp_x^s,$$

where λ is absolutely continuous, and p_x^s is the singular part. Thus,

$$\mathcal{L} = \int (v + p_x f + \lambda h) dt + \int h dp_x^s$$

Then, Condition 4 (Concavity) implies that the first integral is concave in y . If p_x is absolutely continuous on T , the singular part $dp_x^s = 0$, so \mathcal{L} is concave in y . Otherwise, because p_x is increasing and h is concave in y , the second integral is also concave in y . Hence, \mathcal{L} is concave in y .

Step 4: Gateaux differential. Let $\mathcal{Y} := \{y: [0, T] \rightarrow \mathbb{R}_+ \mid y \text{ is increasing}\}$. Taking the Gateaux differential in the direction of $\tilde{y} \in \mathcal{Y}$, we have

$$\partial \mathcal{L}(y; \tilde{y}) = \int_0^T (v_y + p_x f_y) \tilde{y} dt + \int_0^T h_y \tilde{y} dp_x$$

By Condition 1, when p_x and p_y are absolutely continuous, we have $\dot{p}_y = -(v_y + p_x f_y) - h_y \dot{p}_x$. At any point t_0 where p_x or p_y is discontinuous, we have $dp_y(t_0) = -h_y(y(t_0), t_0) dp_x(t_0)$ if y

is continuous at t_0 by Condition 1, and $dp_y(t_0) = dp_x(t_0) = 0$ if y is discontinuous at t_0 by Condition 5 (jump conditions). Because p_x and p_y are piecewise absolutely continuous, we have $dp_y = -(v_y + p_x f_y)dt - h_y dp_x$. Therefore,

$$\partial\mathcal{L}(y; \tilde{y}) = \int_0^T (v_y + p_x f_y) \tilde{y} dt + \int_0^T h_y \tilde{y} dp_x = - \int_0^T \tilde{y}(t) dp_y(t).$$

Using Lebesgue-Stieltjes integration by parts, we have

$$\int_0^T \tilde{y} dp_y = \tilde{y}(T)p_y(T) - \tilde{y}(0)p_y(0) - \int_0^T p_y(t) d\tilde{y}(t).$$

By Condition 3 (transversality), we have $p_y(T) = 0$, so

$$\partial\mathcal{L}(y; \tilde{y}) = \tilde{y}(0)p_y(0) + \int_0^T p_y(t) d\tilde{y}(t).$$

For any $\tilde{y} \in \mathcal{Y}$, because \tilde{y} is increasing (i.e., $d\tilde{y} \geq 0$), Condition 2 implies $p_y(t) \leq 0$ and thus $p_y(t)d\tilde{y}(t) \leq 0$. Condition 3 (transversality) implies $p_y(0) \leq 0$, and since $\tilde{y}(0) \geq 0$, the boundary term $\tilde{y}(0)p_y(0) \leq 0$. Thus, we have $\partial\mathcal{L}(y; \tilde{y}) \leq 0$.

Finally, evaluating at the optimal candidate y^* , we have $\partial\mathcal{L}(y^*; y^*) = 0$ because Condition 2 (monotonicity) implies $p_y(t)dy^*(t) = 0$, and Condition 3 (transversality) implies $y^*(0)p_y(0) = 0$.

Step 5: Apply Luenberger (1969) sufficiency theorem. By Lemma 1 in Luenberger (1969, p. 227), because \mathcal{L} is concave in y , $\partial\mathcal{L}(y^*; y^*) = 0$, and $\partial\mathcal{L}(y^*; \tilde{y}) \leq 0$ for all $\tilde{y} \in \mathcal{Y}$, we have

$$\int_0^T v(h(y^*, t), y^*, t) dt = \mathcal{L}(y^*) \geq \mathcal{L}(\tilde{y}) \geq \int_0^T v(h(\tilde{y}, t), \tilde{y}, t) dt$$

for all $\tilde{y} \in \mathcal{Y}$. Therefore, (x^*, y^*) (where $x^* = h(y^*, t)$) is an optimal solution to the problem. \square

A.2 Optimal Control Method to the Truncated Problem

Proof of Proposition 1. I apply Theorem 1 to the truncated problem [Pt] to show that the price cap allocation with cutoff type γ_t is optimal for [Pt] if conditions (i*) and (ii) hold at γ_t .

Setup of the Hamiltonian. Define $U(\gamma) = \int_{\gamma}^{\gamma_t} q_t(x)dx + \bar{U}$. Rewrite the constraints as:⁹

$$U(\gamma) = -\gamma q_t(\gamma) + b(q_t(\gamma)) - \sigma \quad (\text{A.7})$$

$$\dot{U} = -q_t(\gamma) \quad (\text{A.8})$$

$$\dot{q}_t = \eta(\gamma) \leq 0 \text{ (} q_t \text{ decreasing)} \quad (\text{A.9})$$

$$U(\gamma_t), q_t(\gamma_t) \geq 0 \quad (\text{A.10})$$

$$U(\underline{\gamma}), q_t(\underline{\gamma}) \text{ free.} \quad (\text{A.11})$$

Define the Hamiltonian (Lagrangian) as

$$H = [w(q_t(\gamma), \gamma) - \sigma]f(\gamma) + \lambda(\gamma)[- \gamma q_t(\gamma) + b(q_t(\gamma)) - U(\gamma) - \sigma] - \Lambda(\gamma)q_t(\gamma) - \mu(\gamma)\eta(\gamma) \quad (\text{A.12})$$

where U, q are state variables and η is the control variable; Λ is Hamiltonian multiplier on \dot{U} and μ is Hamiltonian multiplier on \dot{q} ; λ is the Lagrangian multiplier on $U = -\gamma q_t + b(q_t)$.

By the maximum principle, the necessary conditions are

$$-\frac{\partial H}{\partial q} = -w_q f + \lambda(\gamma - b'(q_t)) + \Lambda = \dot{\mu} \quad (\text{A.13})$$

$$-\frac{\partial H}{\partial U} = \lambda = \dot{\Lambda} \quad (\text{A.14})$$

$$\frac{\partial H}{\partial \eta} = \mu \geq 0, \quad \mu(\gamma) = 0 \text{ if } q \text{ is strictly decreasing at } \gamma \quad (\text{A.15})$$

$$\Lambda(\gamma_t) \geq 0, \quad \Lambda(\gamma_t)U(\gamma_t) = 0 \quad (\text{A.16})$$

$$\mu(\gamma_t) \geq 0, \quad \mu(\gamma_t)q_t(\gamma_t) = 0 \quad (\text{A.17})$$

$$\Lambda(\underline{\gamma}) = 0, \quad \mu(\underline{\gamma}) = 0. \quad (\text{A.18})$$

Proposed Multipliers. Recall that $\kappa \equiv \inf_q \{w_{qq}/b''(q)\} = 1 + \inf_q \{v''(q)/\alpha b''(q)\}$. The proposed multipliers are

$$\Lambda(\gamma) = \begin{cases} 0; & \gamma = \underline{\gamma} \\ w_q(\gamma, q_f(\gamma))f(\gamma); & \gamma \in (\underline{\gamma}, \gamma_H(\gamma_t)) \\ A + \kappa(F(\gamma_t) - F(\gamma)); & \gamma \in [\gamma_H(\gamma_t), \gamma_t] \end{cases} \quad (\text{A.19})$$

and

$$A^* = \frac{1}{\gamma_t - b'(q_i(\gamma_t))} \int_{b'(q_i(\gamma_t))}^{\gamma_t} w_q(\tilde{\gamma}, q_i(\gamma_t))f(\tilde{\gamma}) d\tilde{\gamma} \geq 0, \quad (\text{A.20})$$

⁹(IR) is implied by $U(\gamma_t), q_t(\gamma_t) \geq 0$ and $\dot{U} = -q_t(\gamma) \leq 0$.

Therefore, we have $\mu(\gamma) = 0$ in the flexible region $\gamma \in [\underline{\gamma}, \gamma_H(\gamma_t))$, and for all $\gamma \in [\gamma_H(\gamma_t), \gamma_t]$,

$$\mu(\gamma) = \int_{\gamma}^{\gamma_t} w_q(\tilde{\gamma}, q_i(\gamma_t)) f(\tilde{\gamma}) + \kappa(\tilde{\gamma} - b'(q_i(\gamma_t))) f(\tilde{\gamma}) d\tilde{\gamma} - (\gamma_t - \gamma) A^* \geq 0 \quad (\text{A.21})$$

by condition (i*), with equality at $\gamma = b'(q_i(\gamma_t))$ by construction of A^* .¹⁰

Sufficiency. Similar to the previous section, the necessary conditions are sufficient if $\Lambda + \kappa F$ is increasing.¹¹ AB's condition (ii) in Proposition 1 implies $\Lambda + \kappa F$ is increasing on $(\underline{\gamma}, \gamma_H(\gamma_t))$; condition (i) implies $\Lambda + \kappa F$ is increasing at $\gamma_H(\gamma_t)$ if it is interior (by $A \geq 0$ if $\gamma_H(\gamma_t) = \underline{\gamma}$); and $w_q > 0$ implies $\Lambda + \kappa F$ is increasing at $\underline{\gamma}$. \square

A.3 Optimal Control Method to the Untruncated Problem

Proof of Proposition 2. I apply Theorem 1 to the untruncated problem [P].

Setup of the Hamiltonian. Define $U(\gamma) = \int_{\gamma}^{\bar{\gamma}} q(x) dx + \bar{U}$. Rewrite the constraints as:

$$U(\gamma) = -\gamma q(\gamma) + b(q(\gamma)) - \sigma \mathbf{1}(q(\gamma)) \quad (\text{A.23})$$

$$\dot{U} = -q(\gamma) \quad (\text{A.24})$$

$$\dot{q} = \eta(\gamma) \leq 0 \text{ (} q \text{ decreasing)} \quad (\text{A.25})$$

$$U(\bar{\gamma}), q(\bar{\gamma}) \geq 0 \quad (\text{A.26})$$

$$U(\underline{\gamma}), q(\underline{\gamma}) \text{ free.} \quad (\text{A.27})$$

¹⁰AB proposes

$$A = \frac{1}{\gamma_t - \gamma_H(\gamma_t)} \left[\int_{\gamma_H(\gamma_t)}^{\gamma_t} w_q(\gamma, q_i(\gamma_t)) f(\gamma) d\gamma + \kappa(\gamma_H(\gamma_t) - b'(q_i(\gamma_t))) F(\gamma_t) \right] \geq A^*.$$

This also implies $\mu(\gamma) \geq 0$ for all $\gamma \in [\gamma_H(\gamma_t), \gamma_t]$ under their stronger condition (i) because

$$\begin{aligned} \mu(\gamma) &= \int_{\gamma}^{\gamma_t} w_q(\tilde{\gamma}, q_i(\gamma_t)) f(\tilde{\gamma}) + \kappa(\tilde{\gamma} - b'(q_i(\gamma_t))) f(\tilde{\gamma}) d\tilde{\gamma} - (\gamma_t - \gamma) A \\ &= \int_{\gamma}^{\gamma_t} w_q(\tilde{\gamma}, q_i(\gamma_t)) f(\tilde{\gamma}) d\tilde{\gamma} + \kappa(\tilde{\gamma} - b'(q_i(\gamma_t))) (F(\gamma_t) - F(\tilde{\gamma})) - (\gamma_t - \gamma) A \geq 0. \end{aligned} \quad (\text{A.22})$$

¹¹See also Remark 7 for an alternative approach à la Kartik et al. (2021) that incorporates a penalty term into the objective and relax the equality constraint (IC-Env) to a concave inequality constraint.

Define the Hamiltonian as

$$H = [w(q(\gamma), \gamma)]f(\gamma) + \lambda(\gamma)[- \gamma q(\gamma) + b(q(\gamma)) - U(\gamma) - \sigma \mathbf{1}(q(\gamma))] - \Lambda(\gamma)q(\gamma) - \mu(\gamma)\eta(\gamma). \quad (\text{A.28})$$

By the maximum principle (see Footnote 14), the necessary conditions are

$$-\frac{\partial H}{\partial q} = -w_q f + \lambda(\gamma - b'(q)) + \Lambda = \dot{\mu} \quad (\text{A.29})$$

$$-\frac{\partial H}{\partial U} = \lambda = \dot{\Lambda} \quad (\text{A.30})$$

$$\frac{\partial H}{\partial \eta} = \mu \geq 0, \quad \mu(\gamma) = 0 \text{ if } q \text{ is strictly decreasing at } \gamma \quad (\text{A.31})$$

$$\Lambda(\bar{\gamma}) \geq 0, \quad \Lambda(\bar{\gamma})U(\bar{\gamma}) = 0 \quad (\text{A.32})$$

$$\mu(\bar{\gamma}) \geq 0, \quad \mu(\bar{\gamma})q(\bar{\gamma}) = 0 \quad (\text{A.33})$$

$$\Lambda(\underline{\gamma}) = 0, \quad \mu(\underline{\gamma}) = 0. \quad (\text{A.34})$$

At the cutoff $\gamma_t \in (\underline{\gamma}, \bar{\gamma}]$,¹² where the state variable q jumps, the following conditions must hold (see, e.g., Seierstad and Sydsæter (1987, Theorem 7, pp. 196–7))

$$\Lambda(\gamma_{t+}) - \Lambda(\gamma_{t-}) = 0 \quad (\text{A.35})$$

$$H(\gamma_{t+}) - H(\gamma_{t-}) = -(w(q_t, \gamma_t))f(\gamma_t) + \Lambda(\gamma_{t-})q_i(\gamma_t) \begin{cases} = 0, & \text{if } \gamma_t \in (\underline{\gamma}, \bar{\gamma}) \\ \leq 0, & \text{if } \gamma_t = \bar{\gamma} \end{cases} \quad (\text{A.36})$$

Proposed Multipliers. The proposed multipliers are¹³

$$\Lambda(\gamma) = \begin{cases} 0; & \gamma = \underline{\gamma} \\ w_q(\gamma, q_f(\gamma))f(\gamma); & \gamma \in (\underline{\gamma}, \gamma_H(\gamma_t)) \\ A^* + \kappa(F(\gamma_t) - F(\gamma)); & \gamma \in [\gamma_H(\gamma_t), \bar{\gamma}] \end{cases} \quad (\text{A.37})$$

where $\kappa \equiv \inf_q \{w_{qq}(q, \gamma)/b''(q)\} = 1 + \inf_q \{v''(q)/\alpha b''(q)\}$, and

$$A^* = \frac{1}{\gamma_t - b'(q_i(\gamma_t))} \int_{\gamma_H(\gamma_t)}^{\gamma_t} w_q(\gamma, q_i(\gamma_t)) dF(\gamma) \geq 0, \quad (\text{A.38})$$

which is smaller than AB's multiplier A .

¹²By Assumption 1 in AB, $\gamma_t \leq \bar{\gamma}$ because the principal-optimal quantity for $\bar{\gamma}$ is higher than $q_i(\bar{\gamma})$ (see their Lemma 1). Additionally, $\gamma_t > \underline{\gamma}$ because $\gamma_t = \underline{\gamma}$ is dominated by $\gamma_t = \bar{\gamma}$.

¹³By convention, $F(\gamma) = 0$ for all $\gamma \leq \underline{\gamma}$.

The cutoff type γ_t , determined by the jump condition (A.36), is given by

$$(w(\gamma_t, q_i(\gamma_t)))f(\gamma_t) = A^* \cdot q_i(\gamma_t) \text{ if } \gamma_t < \bar{\gamma}. \quad (\text{A.39})$$

Otherwise, if $(w(\gamma_t, q_i(\gamma_t)))f(\gamma_t) - A^* \cdot q_i(\gamma_t) > 0$, the cutoff type is $\gamma_t = \bar{\gamma}$.

$$\mu(\gamma) = \begin{cases} \int_{\gamma}^{\gamma_t} s(\tilde{\gamma}) d\tilde{\gamma} - (\gamma_t - \gamma)A^* \geq 0, & \gamma \in [\gamma_H(\gamma_t), \bar{\gamma}] \\ 0, & \gamma \in [\underline{\gamma}, \gamma_H(\gamma_t)] \end{cases} \quad (\text{A.40})$$

As in the truncated problem, the inequality $\mu(\gamma) \geq 0$ is implied by condition (i*).

Sufficiency. Because $\Lambda + \kappa F$ is increasing, sufficiency follows similarly as in the previous section. \square

A.4 Proofs of Sections 3–4

Lemma A.1. $L(\gamma_t + |\gamma_t|) \geq L(\gamma_t - |\gamma_t|)$ for all $\gamma_t \in (\underline{\gamma}, \bar{\gamma})$. The equality holds if and only if $w_{qq}(q, \gamma_t) + \kappa b''(q) = 0$ for almost every $q \in (0, q_i(\gamma_t))$.

Proof of Lemma A.1. By definition,

$$\begin{aligned} L(\gamma_t + |\gamma_t|) &= w_q(\gamma_t, 0) f(\gamma_t) + \kappa f(\gamma_t) (\gamma_t - b'(0)) \\ L(\gamma_t - |\gamma_t|) &= w_q(\gamma_t, q_i(\gamma_t)) f(\gamma_t) + \kappa f(\gamma_t) (\gamma_t - b'(q_i(\gamma_t))) \end{aligned}$$

Therefore,

$$\begin{aligned} L(\gamma_t + |\gamma_t|) - L(\gamma_t - |\gamma_t|) &= f(\gamma_t) \left(w_q(\gamma_t, 0) - w_q(\gamma_t, q_i(\gamma_t)) + \kappa (b'(q_i(\gamma_t)) - b'(0)) \right) \\ &= -f(\gamma_t) \int_0^{q_i(\gamma_t)} \left(w_{qq}(\gamma_t, q) - \kappa b''(q) \right) dq, \end{aligned}$$

Hence, $L(\gamma_t + |\gamma_t|) \geq L(\gamma_t - |\gamma_t|)$ follows from

$$w_{qq}(\gamma_t, q) - \kappa b''(q) \leq 0 \quad \text{for a.e. } q \in (0, q_i(\gamma_t))$$

because $\kappa = \inf_q \{w_{qq}(\gamma, q)/b''(q)\}$ and $b''(q) < 0$. Moreover, the equality $L(\gamma_t + |\gamma_t|) = L(\gamma_t - |\gamma_t|)$ holds if and only if $w_{qq}(\gamma_t, q) - \kappa b''(q) = 0$ for almost every $q \in (0, q_i(\gamma_t))$. \square

Proof Sketch of Corollary 2.2. A unimodal $s(\gamma)$ (with a convex-concave S) with peak $\hat{\gamma} \in (\underline{\gamma}, \bar{\gamma})$ satisfies conditions (S) and (ii) at some $\gamma \in (\underline{\gamma}, \hat{\gamma})$. A decreasing $s(\gamma)$ (with a concave

S) satisfies condition (S) at some γ such that $b'_i(q_i(\gamma)) \leq \underline{\gamma}$ and thus satisfies condition (ii) vacuously. An increasing $s(\gamma)$ (with a convex S) satisfies conditions (S) and (ii) at $\gamma = \bar{\gamma}$. \square

Proof of Corollary 3.1. For $\tilde{s}(\gamma) = (\beta/\alpha)f(\gamma) + F(\gamma)$, we have

$$\tilde{s}'(\gamma) = f(\gamma) \left(\frac{\beta}{\alpha} \frac{f'(\gamma)}{f(\gamma)} + 1 \right).$$

If f is log-concave, then $\frac{f'(\gamma)}{f(\gamma)}$ is decreasing in γ , and $\tilde{s}'(\gamma)$ changes sign at most once from positive to negative, so \tilde{s} is unimodal. Conditions 2 and 3 are the conditions for $\tilde{s}'(\gamma) \geq 0$ and $\tilde{s}'(\gamma) \leq 0$, respectively. Hence, the corollary follows from Proposition 3. \square

Proof of Corollary 3.2. For $\tilde{s}(\gamma) = (c\gamma - d(\gamma))f(\gamma) + cF(\gamma)$, we have

$$\tilde{s}'(\gamma) = f(\gamma) \left[(2c - 1) + (c - 1) (\gamma - P_0) \frac{f'(\gamma)}{f(\gamma)} \right].$$

The assumptions $\rho < 1$ and $\bar{\gamma} \leq P_0$ imply $c < 1$ and $\gamma \leq P_0$. Therefore, $\tilde{s}'(\gamma) \geq 0$ if and only if

$$\frac{f'(\gamma)}{f(\gamma)} \geq \frac{1 - 2c}{(1 - c)(P_0 - \gamma)},$$

which is conditions 2 (and 3 after reversing the sign) after substituting $c = 1 - \frac{1}{\alpha} \frac{1}{1 - \epsilon}$. To show condition 1, if $\epsilon \geq 1 - \frac{2}{\alpha}$, then $c \leq 1/2$, so the right-hand side is positive and strictly increasing in γ when $\gamma \leq P_0$. If f is log-concave, then $\frac{f'(\gamma)}{f(\gamma)}$ is decreasing in γ . Therefore, $\tilde{s}'(\gamma)$ changes sign at most once from positive to negative, and thus \tilde{s} is unimodal. Hence, the corollary follows from Proposition 3. \square

Proof of Corollary 3.3. For $\tilde{s}(\gamma) = (c\gamma - d(\gamma))f(\gamma) + cF(\gamma)$, we have

$$\tilde{s}'(\gamma) = f(\gamma) \left[(2c - 1) + (c - 1) (\gamma - P_0) \frac{f'(\gamma)}{f(\gamma)} \right]$$

The assumptions $\rho > 1$ and $\underline{\gamma} \geq P_0$ imply $c = 1 - \frac{1}{\alpha} \frac{1}{1 - \epsilon} > 1$ and $\gamma \geq P_0$. Therefore, $\tilde{s}'(\gamma) \geq 0$ if and only if

$$\frac{f'(\gamma)}{f(\gamma)} \geq -\frac{2c - 1}{(c - 1)(\gamma - P_0)},$$

which is conditions 2 (and 3 after reversing the sign) after substituting $c = 1 - \frac{1}{\alpha} \frac{1}{1 - \epsilon}$. To show condition 1, note that when $c > 1$ and $\gamma \geq P_0$, right-hand side is positive and strictly increasing in γ . If f is log-concave, then $\frac{f'(\gamma)}{f(\gamma)}$ is decreasing in γ . Therefore, $\tilde{s}'(\gamma)$ changes

sign at most once from positive to negative, and thus \tilde{s} is unimodal. Hence, the corollary follows from Proposition 3. \square

Proof of Corollary 3.4. For $\tilde{s}(\gamma) = (1 - \gamma)f(\gamma)/2 + F(\gamma)/2$, we have

$$\tilde{s}'(\gamma) = (1 - \gamma)f'(\gamma)/2 \stackrel{\text{sign}}{=} f'(\gamma)$$

because $\gamma \leq 1$. Thus, $\tilde{s}(\gamma)$ is increasing (decreasing) if and only if $f(\gamma)$ is increasing (decreasing). Hence, the corollary follows from Proposition 3. \square

Appendix B Applications to delegation problems

In this section, I apply Theorem 1 to delegation problems without participation constraint (Amador and Bagwell, 2013). I focus on the case without money burning.

The maximization problem is

$$\max_{\pi: \Gamma \rightarrow \Pi} \int w(\gamma, \pi(\gamma)) dF(\gamma) \quad \text{subject to:} \quad (\text{B.1})$$

$$\gamma\pi(\gamma) + b(\pi(\gamma)) = \int_{\underline{\gamma}}^{\gamma} \pi(\tilde{\gamma}) d\tilde{\gamma} + \underline{U} \quad \text{for all } \gamma \in \Gamma, \quad (\text{B.2})$$

$$\pi \text{ increasing}, \quad (\text{B.3})$$

where $\underline{U} \equiv \underline{\gamma}\pi(\underline{\gamma}) + b(\pi(\underline{\gamma}))$.

Define $\pi_f(\gamma) = \arg \max_{\pi} \{\gamma\pi + b(\pi)\}$ as the flexible allocation. I look for conditions for the following interval delegation allocation to be optimal:

$$\pi(\gamma) = \begin{cases} \pi_f(\gamma_L); & \gamma \in [\underline{\gamma}, \gamma_L], \\ \pi_f(\gamma); & \gamma \in (\gamma_L, \gamma_H), \\ \pi_f(\gamma_H); & \gamma \in [\gamma_H, \bar{\gamma}] \end{cases} \quad (\text{B.4})$$

Setup of the Hamiltonian. Define $U = \int_{\underline{\gamma}}^{\gamma} \pi(\tilde{\gamma})d\tilde{\gamma} + \underline{U}$. Rewrite the constraints as

$$\gamma\pi(\gamma) + b(\pi(\gamma)) = U \quad (\text{B.5})$$

$$\dot{U} = \pi \quad (\text{B.6})$$

$$\dot{\pi} = \eta \geq 0 \text{ (}\pi \text{ increasing)} \quad (\text{B.7})$$

$$U(\underline{\gamma}), \pi(\underline{\gamma}) \text{ free,} \quad (\text{B.8})$$

$$U(\bar{\gamma}), \pi(\bar{\gamma}) \text{ free} \quad (\text{B.9})$$

Set up the Hamiltonian as

$$H = w(\gamma, \pi)f(\gamma) + \lambda(\gamma\pi(\gamma) + b(\pi(\gamma)) - U) + \Lambda\pi + \mu\eta \quad (\text{B.10})$$

where π, U are the state variable and η is the control variable; Λ is the Hamiltonian multiplier (costate) on \dot{U} , μ is the costate for $\dot{\pi}$, and λ is the Lagrangian multiplier on $\gamma\pi(\gamma) + b(\pi(\gamma)) = U$.

By the maximum principle, the necessary conditions are¹⁴

$$-\frac{\partial H}{\partial \pi} = -(w_{\pi}f + \lambda(\gamma + b'(\pi)) + \Lambda) = \dot{\mu} \quad (\text{B.11})$$

$$-\frac{\partial H}{\partial U} = \lambda = \dot{\Lambda} \quad (\text{B.12})$$

$$\frac{\partial H}{\partial \eta} = \mu \leq 0, \quad \mu(\gamma) = 0 \text{ if } \pi \text{ is strictly decreasing at } \gamma \quad (\text{B.13})$$

$$\Lambda(\underline{\gamma}) = 0, \quad \mu(\underline{\gamma}) = 0 \quad (\text{B.14})$$

$$\Lambda(\bar{\gamma}) = 0, \quad \mu(\bar{\gamma}) = 0 \quad (\text{B.15})$$

Proposed Multipliers. Define $\kappa = \inf_{(\gamma, \pi) \in \Gamma \times \Pi} \left\{ \frac{w_{\pi\pi}(\gamma, \pi)}{b''(\pi)} \right\}$. The proposed multipliers are

$$\Lambda(\gamma) = \begin{cases} \kappa(1 - F(\gamma)), & \gamma \in [\gamma_H, \bar{\gamma}], \\ -w_{\pi}(\gamma, \pi_f(\gamma))f(\gamma), & \gamma \in (\gamma_L, \gamma_H), \\ -\kappa F(\gamma), & \gamma \in [\underline{\gamma}, \gamma_L] \end{cases} \quad (\text{B.16})$$

¹⁴The formulation is by Hellwig (2010, Theorem 4.1), which does not require the state variable π to be continuous. See also Léonard and Long (1992, Theorem 10.2.1) for the necessary conditions with inequality and equality constraints.

$$\mu(\gamma) = \begin{cases} \int_{\gamma}^{\bar{\gamma}} (w_{\pi}(\tilde{\gamma}, \pi_f(\gamma_H))f(\tilde{\gamma}) - \kappa\tilde{\gamma}f(\tilde{\gamma}) + \kappa(1 - F(\tilde{\gamma}))) d\tilde{\gamma} \leq 0, & \gamma \in [\gamma_H, \bar{\gamma}], \\ 0, & \gamma \in (\gamma_L, \gamma_H), \\ - \int_{\underline{\gamma}}^{\gamma} (w_{\pi}(\tilde{\gamma}, \pi_f(\gamma_L))f(\tilde{\gamma}) - \kappa\tilde{\gamma}f(\tilde{\gamma}) + \kappa F(\tilde{\gamma})) d\tilde{\gamma} \leq 0, & \gamma \in [\underline{\gamma}, \gamma_L] \end{cases} \quad (\text{B.17})$$

Now we are ready to show the following conditions are sufficient for interval delegation to be optimal (Proposition 1 in Amador and Bagwell (2013)).

(c1) $\kappa F(\gamma) - w_{\pi}(\gamma, \pi_f(\gamma)) f(\gamma)$ is increasing for all $\gamma \in [\gamma_L, \gamma_H]$.

(c2) If $\gamma_H < \bar{\gamma}$,

$$(\gamma - \gamma_H) \kappa \geq \int_{\gamma}^{\bar{\gamma}} w_{\pi}(\tilde{\gamma}, \pi_f(\gamma_H)) \frac{f(\tilde{\gamma})}{1 - F(\tilde{\gamma})} d\tilde{\gamma}, \quad \forall \gamma \in [\gamma_H, \bar{\gamma}]$$

with equality at γ_H .

(c2') If $\gamma_H = \bar{\gamma}$, $w_{\pi}(\bar{\gamma}, \pi_f(\bar{\gamma})) \geq 0$.

(c3) If $\gamma_L > \underline{\gamma}$,

$$(\gamma - \gamma_L) \kappa \leq \int_{\underline{\gamma}}^{\gamma} w_{\pi}(\tilde{\gamma}, \pi_f(\gamma_L)) \frac{f(\tilde{\gamma})}{F(\tilde{\gamma})} d\tilde{\gamma}, \quad \forall \gamma \in [\underline{\gamma}, \gamma_L]$$

with equality at γ_L .

(c3') If $\gamma_L = \underline{\gamma}$, $w_{\pi}(\underline{\gamma}, \pi_f(\underline{\gamma})) \leq 0$.

The inequality $\mu(\gamma) \leq 0$ follows from their assumption (c2) [resp. (c3)] if γ_H [resp. γ_L] is interior. If γ_H or γ_L is at the boundary, the corresponding inequality is satisfied trivially.

Sufficiency. By Theorem 1, the necessary conditions above are sufficient if (1) H is concave in the state and control variables (π, U, η) for the proposed multipliers (λ, μ, Λ) and (2) Λ is increasing. Because H is linear in η and U , (1) holds if $H_{\pi\pi} = w_{\pi\pi}f + \lambda b''(\pi) \leq 0$. Because $\kappa = \inf_{\pi, \gamma} \{w_{\pi\pi}(\pi, \gamma)/b''(\pi)\}$ and $\Lambda' = \lambda$, (1) and (2) are both satisfied if $\Lambda + \kappa F$ is increasing.

Therefore, their assumption (c1) implies $\Lambda + \kappa F$ is increasing on (γ_L, γ_H) , and (c2') and (c3') implies $\Lambda + \kappa F$ is increasing at $\underline{\gamma}$ and $\bar{\gamma}$, respectively (if γ_H or γ_L are at the boundary).

Remark 7. One can also incorporate a penalty term $-\kappa(\gamma\pi(\gamma) + b(\pi(\gamma)) - U)$ into the objective function and relax the equality constraint (IC-Env) to a concave inequality constraint $\gamma\pi(\gamma) + b(\pi(\gamma)) - U \geq 0$ (IC-Env') à la Kartik et al. (2021). For this relaxed problem, the necessary conditions implied by the maximum principle remain unchanged

after redefining the Lagrangian multiplier on (IC-Env') as $\tilde{\lambda} = \lambda + \kappa f$. Then, dual feasibility of (IC-Env') requires $\tilde{\lambda} = \lambda + \kappa f \geq 0$, which is satisfied if $\Lambda + \kappa F$ is increasing, which is the same condition for sufficiency (or concavity of the Hamiltonian).

Then, the problem becomes an optimal control problem with concave state inequality constraints, and the standard sufficiency theorem (e.g., Léonard and Long (1992, Theorem 6.5.3), Seierstad and Sydsæter (1987, Theorem 1 in Chapter 5)) is applicable. In particular, jumps in costate variable Λ (which occurs at junction points between intervals on which (IC-Env') binds and does not bind) needs to be nonnegative because $\frac{\partial}{\partial U}(\gamma\pi(\gamma) + b(\pi(\gamma)) - U) < 0$, which is also satisfied if $\Lambda + \kappa F$ is increasing.

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